Supplemental Material: Bifurcation analysis of twisted liquid crystal bilayers

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This Supplemental Material (SM) explicitly derives the matrices $\mathbf{Q}^{(r)}$, $\mathbf{H}^{(r)}_{c,s}$ and $\mathbf{F}^{(r)}_{c,s}$ (for $r = 1, 2$), that are appearing in the analytical solution of Section 5 of the main article. We first expand the governing equations of the main text (A5.5) [1]

$$k_1^{(r)}(\Delta n_{2,22} + \Delta n_{3,32}) + k_2^{(r)}(\Delta n_{2,33} - \Delta n_{3,23}) + k_3^{(r)} \Delta n_{2,11} = 0, \tag{1}$$

$$k_1^{(r)}(\Delta n_{2,23} + \Delta n_{3,33}) + k_2^{(r)}(\Delta n_{3,22} - \Delta n_{2,32}) + k_3^{(r)} \Delta n_{3,11} + \frac{C_\perp^{(r)}}{C_b^{(r)}} d_0 \left[C_\perp^{(r)} \frac{d_0}{\varepsilon_0} \Delta n_3 + \Delta\varphi_{,1} \right] = 0, \tag{2}$$

$$\Delta\varphi_{,11} + \Delta\varphi_{,22} + \Delta\varphi_{,33} + \frac{C_\perp^{(r)}}{C_b^{(r)}} \left[\Delta\varphi_{,11} + C_\perp^{(r)} \frac{d_0}{\varepsilon_0} \Delta n_{3,1} \right] = 0. \tag{3}$$

Substituting the eigenmode (A5.8) to the above, we obtain

$$k_1^{(r)}(\omega_2^2 \Delta\mathcal{N}_2^{(r)} + \omega_2 \mathcal{N}_{3,3}^{(r)}) - k_2^{(r)}(\Delta\mathcal{N}_{2,33}^{(r)} + \omega_2 \Delta\mathcal{N}_{3,3}^{(r)}) + k_3^{(r)} \omega_1^2 \Delta\mathcal{N}_2^{(r)} = 0, \tag{4}$$

$$\begin{aligned} &k_1^{(r)}(\omega_2 \Delta\mathcal{N}_{2,3}^{(r)} + \Delta\mathcal{N}_{3,33}^{(r)}) - \\ &k_2^{(r)}(\omega_2^2 \Delta\mathcal{N}_3^{(r)} + \omega_2 \Delta\mathcal{N}_{2,3}^{(r)}) - k_3^{(r)} \omega_1^2 \Delta\mathcal{N}_3^{(r)} + C^{(r)} d_0 \left[C_\perp^{(r)} \frac{d_0}{\varepsilon_0} \Delta\mathcal{N}_3^{(r)} + \omega_1 \Delta\Phi^{(r)} \right] = 0, \end{aligned} \tag{5}$$

$$-\omega_1^2 \Delta\Phi^{(r)} - \omega_2^2 \Delta\Phi^{(r)} + \Delta\Phi_{,33}^{(r)} - C^{(r)} \left[\omega_1^2 \Delta\Phi^{(r)} + C_\perp^{(r)} \frac{d_0}{\varepsilon_0} \omega_1 \Delta\mathcal{N}_3^{(r)} \right] = 0, \tag{6}$$

where $C^{(r)} = C_\perp^{(r)}/C_b^{(r)}$. One can show that the above set of ordinary differential equations admit a general solution of the form (A5.9). The above, upon substitution of (A5.9) yields a set of three homogeneous algebraic equations (A5.10), where $\mathbf{Q}^{(r)}$ is given by

$$\mathbf{Q}^{(r)} \equiv \begin{bmatrix} k_1^{(r)}\omega_2^2 + k_3^{(r)}\omega_1^2 - k_2^{(r)}\rho^2 & (k_1^{(r)} - k_2^{(r)})\omega_2\rho & 0 \\ (k_1^{(r)} - k_2^{(r)})\omega_2\rho & -k_2^{(r)}\omega_2^2 - k_3^{(r)}\omega_1^2 + k_1^{(r)}\rho^2 + C^{(r)}C_\perp^{(r)} d_0^2/\varepsilon_0 & C^{(r)} d_0 \omega_1 \\ 0 & C^{(r)} C_\perp^{(r)} d_0 \omega_1/\varepsilon_0 & (1 + C^{(r)})\omega_1^2 + \omega_2^2 - \rho^2 \end{bmatrix}. \tag{7}$$

To obtain a non-trivial solution of (A5.10), the condition $\det[\mathbf{Q}^{(r)}] = 0$ must be satisfied. This condition leads to a characteristic bi-cubic polynomial in ρ given by (A5.11). Subsequently, the eigenmodes $\Delta\widehat{\mathcal{N}}_1$ and $\Delta\widehat{\mathcal{N}}_2$ are given in (A5.12).

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[1] henceforth the letter A will be used to denote the corresponding equation number in the main article.

Then, it only remains to calculate the unknown amplitudes $\boldsymbol{\Xi}$ from the interface conditions (A5.6), (A5.7) and the boundary conditions (A4.3) and (A5.2). For that, we write the interface conditions $(A5.6)_1$ and $(A5.7)_1$ at $x_3 = 0$, such that

$$[\![\Delta\mathcal{N}_2]\!] = 0 \quad \Longrightarrow \quad \sum_{I=1}^{3}(\xi_I^{c,(2)}\Delta\widehat{\mathcal{N}}_2^{I,(2)} - \xi_I^{c,(1)}\Delta\widehat{\mathcal{N}}_2^{I,(1)}) = 0, \tag{8}$$

$$[\![\Delta\mathcal{N}_3]\!] = 0 \quad \Longrightarrow \quad \sum_{I=1}^{3}(\xi_I^{s,(2)}\Delta\widehat{\mathcal{N}}_3^{I,(2)} - \xi_I^{s,(1)}\Delta\widehat{\mathcal{N}}_3^{I,(1)}) = 0, \tag{9}$$

$$[\![\Delta\Phi]\!] = 0 \quad \Longrightarrow \quad \sum_{I=1}^{3}(\xi_I^{s,(2)} - \xi_I^{s,(1)}) = 0. \tag{10}$$

Similarly, the interface conditions $(A5.6)_2$ and $(A5.7)_3$ are now obtained explicitly as

$$\left[\!\left[\mathcal{L}_{23kl}^{\nabla n\nabla n,(r)}\Delta n_{k,l}\right]\!\right] = 0 \quad \Longrightarrow \quad \left[\!\left[k_2^{(r)}(\Delta n_{2,3} - \Delta n_{3,2})\right]\!\right] = 0$$

$$\Longrightarrow \quad \sum_{I=1}^{3}\left[k_2^{(2)}\xi_I^{s,(2)}\left(\rho_I^{(2)}\Delta\widehat{\mathcal{N}}_2^{I,(2)} + \omega_2\Delta\widehat{\mathcal{N}}_3^{I,(2)}\right) - k_2^{(1)}\xi_I^{s,(1)}\left(\rho_I^{(1)}\Delta\widehat{\mathcal{N}}_2^{I,(1)} + \omega_2\Delta\widehat{\mathcal{N}}_3^{I,(1)}\right)\right] = 0, \tag{11}$$

$$\left[\!\left[\mathcal{L}_{33kl}^{\nabla n\nabla n,(r)}\Delta n_{k,l}\right]\!\right] = 0 \quad \Longrightarrow \quad \left[\!\left[k_1^{(r)}(\Delta n_{2,2} + \Delta n_{3,3})\right]\!\right] = 0$$

$$\Longrightarrow \quad \sum_{I=1}^{3}\left[k_1^{(2)}\xi_I^{c,(2)}\left(\omega_2\Delta\widehat{\mathcal{N}}_2^{I,(2)} + \rho_I^{(2)}\Delta\widehat{\mathcal{N}}_3^{I,(2)}\right) - k_1^{(1)}\xi_I^{c,(1)}\left(\omega_2\Delta\widehat{\mathcal{N}}_2^{I,(1)} + \rho_I^{(1)}\Delta\widehat{\mathcal{N}}_3^{I,(1)}\right)\right] = 0, \tag{12}$$

$$\left[\!\left[\mathcal{L}_{3j}^{n\nabla\varphi,(r)}\Delta n_j + \mathcal{L}_{3j}^{\nabla\varphi\nabla\varphi,(r)}\Delta\varphi_{,j}\right]\!\right] = 0 \quad \Longrightarrow \quad \left[\!\left[\frac{\varepsilon_0}{C_\perp^{(r)}}\Delta\Phi_{,3}\right]\!\right] = 0$$

$$\Longrightarrow \quad \sum_{I=1}^{3}\left(\frac{\varepsilon_0}{C_\perp^{(2)}}\rho_I^{(2)}\xi_I^{c,(2)} - \frac{\varepsilon_0}{C_\perp^{(1)}}\rho_I^{(1)}\xi_I^{c,(1)}\right) = 0. \tag{13}$$

Equations (8), (12) and (13) are now re-written in the following matrix form:

$$\mathbf{H}_c^{(2)}\boldsymbol{\xi}^{c,(2)} = \mathbf{H}_c^{(1)}\boldsymbol{\xi}^{c,(1)}, \tag{14}$$

where

$$\mathbf{H}_c^{(r)} \equiv \begin{bmatrix} \Delta\widehat{\mathcal{N}}_2^{1,(r)} & \Delta\widehat{\mathcal{N}}_2^{2,(r)} & \Delta\widehat{\mathcal{N}}_2^{3,(r)} \\ k_1^{(r)}\left(\omega_2\Delta\widehat{\mathcal{N}}_2^{1,(r)} + \rho_1^{(r)}\Delta\widehat{\mathcal{N}}_3^{1,(r)}\right) & k_1^{(r)}\left(\omega_2\Delta\widehat{\mathcal{N}}_2^{2,(r)} + \rho_2^{(r)}\Delta\widehat{\mathcal{N}}_3^{2,(r)}\right) & k_1^{(r)}\left(\omega_2\Delta\widehat{\mathcal{N}}_2^{3,(r)} + \rho_3^{(r)}\Delta\widehat{\mathcal{N}}_3^{3,(r)}\right) \\ \varepsilon_0\rho_1^{(r)}/C_\perp^{(r)} & \varepsilon_0\rho_2^{(r)}/C_\perp^{(r)} & \varepsilon_0\rho_3^{(r)}/C_\perp^{(r)} \end{bmatrix},$$

with $r = 1, 2$. Similarly, (9 – 11) are now expressed as

$$\mathbf{H}_s^{(2)}\boldsymbol{\xi}^{s,(2)} = \mathbf{H}_s^{(1)}\boldsymbol{\xi}^{s,(1)}, \tag{15}$$

where

$$\mathbf{H}_s^{(r)} \equiv \begin{bmatrix} \Delta\widehat{\mathcal{N}}_3^{1,(r)} & \Delta\widehat{\mathcal{N}}_3^{2,(r)} & \Delta\widehat{\mathcal{N}}_3^{3,(r)} \\ 1 & 1 & 1 \\ k_2^{(r)}\left(\rho_1^{(r)}\Delta\widehat{\mathcal{N}}_2^{1,(r)} + \omega_2\Delta\widehat{\mathcal{N}}_3^{1,(r)}\right) & k_2^{(r)}\left(\rho_2^{(r)}\Delta\widehat{\mathcal{N}}_2^{2,(r)} + \omega_2\Delta\widehat{\mathcal{N}}_3^{2,(r)}\right) & k_2^{(r)}\left(\rho_3^{(r)}\Delta\widehat{\mathcal{N}}_2^{3,(r)} + \omega_2\Delta\widehat{\mathcal{N}}_3^{3,(r)}\right) \end{bmatrix}.$$

The boundary conditions $(A4.3)_3$ and $(A5.2)_3$ lead to

$$\Delta\mathcal{N}_2 = 0 \quad \Longrightarrow \quad \sum_{I=1}^{3}\{\xi_I^{s,(2)}\sinh(\rho_I^{(2)}\ell_3^{(2)}) + \xi_I^{c,(2)}\cosh(\rho_I^{(2)}\ell_3^{(2)})\}\Delta\widehat{\mathcal{N}}_2^{I,(2)} = 0, \tag{16}$$

$$\Delta\mathcal{N}_3 = 0 \quad \Longrightarrow \quad \sum_{I=1}^{3}\{\xi_I^{s,(2)}\cosh(\rho_I^{(2)}\ell_3^{(2)}) + \xi_I^{c,(2)}\sinh(\rho_I^{(2)}\ell_3^{(2)})\}\Delta\widehat{\mathcal{N}}_3^{I,(2)} = 0, \tag{17}$$

$$\Delta\Phi = 0 \quad \Longrightarrow \quad \sum_{I=1}^{3}\{\xi_I^{s,(2)}\cosh(\rho_I^{(2)}\ell_3^{(2)}) + \xi_I^{c,(2)}\sinh(\rho_I^{(2)}\ell_3^{(2)})\} = 0. \tag{18}$$

The above equations can be expressed in matrix form as

$$\mathbf{F}_c^{(2)}\boldsymbol{\xi}^{c,(2)} = \mathbf{F}_s^{(2)}\boldsymbol{\xi}^{s,(2)}, \tag{19}$$

where

$$\mathbf{F}_c^{(2)} \equiv \begin{bmatrix} \cosh(\rho_1^{(2)}\ell_3^{(2)})\Delta\widehat{\mathcal{N}}_2^{1,(2)} & \cosh(\rho_2^{(2)}\ell_3^{(2)})\Delta\widehat{\mathcal{N}}_2^{2,(2)} & \cosh(\rho_3^{(2)}\ell_3^{(2)})\Delta\widehat{\mathcal{N}}_2^{3,(2)} \\ \sinh(\rho_1^{(2)}\ell_3^{(2)})\Delta\widehat{\mathcal{N}}_3^{1,(2)} & \sinh(\rho_2^{(2)}\ell_3^{(2)})\Delta\widehat{\mathcal{N}}_3^{2,(2)} & \sinh(\rho_3^{(2)}\ell_3^{(2)})\Delta\widehat{\mathcal{N}}_3^{3,(2)} \\ \sinh(\rho_1^{(2)}\ell_3^{(2)}) & \sinh(\rho_2^{(2)}\ell_3^{(2)}) & \sinh(\rho_3^{(2)}\ell_3^{(2)}) \end{bmatrix},$$

and

$$\mathbf{F}_s^{(2)} \equiv \begin{bmatrix} -\sinh(\rho_1^{(2)}\ell_3^{(2)})\Delta\widehat{\mathcal{N}}_2^{1,(2)} & -\sinh(\rho_2^{(2)}\ell_3^{(2)})\Delta\widehat{\mathcal{N}}_2^{2,(2)} & -\sinh(\rho_3^{(2)}\ell_3^{(2)})\Delta\widehat{\mathcal{N}}_2^{3,(2)} \\ -\cosh(\rho_1^{(2)}\ell_3^{(2)})\Delta\widehat{\mathcal{N}}_3^{1,(2)} & -\cosh(\rho_2^{(2)}\ell_3^{(2)})\Delta\widehat{\mathcal{N}}_3^{2,(2)} & -\cosh(\rho_3^{(2)}\ell_3^{(2)})\Delta\widehat{\mathcal{N}}_3^{3,(2)} \\ -\cosh(\rho_1^{(2)}\ell_3^{(2)}) & -\cosh(\rho_2^{(2)}\ell_3^{(2)}) & -\cosh(\rho_3^{(2)}\ell_3^{(2)}) \end{bmatrix}.$$

Similarly, from $(A4.3)_1$ and $(A5.2)_1$, we obtain

$$\mathbf{F}_c^{(1)}\boldsymbol{\xi}^{c,(1)} = \mathbf{F}_s^{(1)}\boldsymbol{\xi}^{s,(1)}, \tag{20}$$

where the matrices $\mathbf{F}_c^{(1)}$ and $\mathbf{F}_s^{(1)}$ are given by

$$\mathbf{F}_c^{(1)} \equiv \begin{bmatrix} \cosh(\rho_1^{(1)}\ell_3^{(1)})\Delta\widehat{\mathcal{N}}_2^{1,(1)} & \cosh(\rho_2^{(1)}\ell_3^{(1)})\Delta\widehat{\mathcal{N}}_2^{2,(1)} & \cosh(\rho_3^{(1)}\ell_3^{(1)})\Delta\widehat{\mathcal{N}}_2^{3,(1)} \\ -\sinh(\rho_1^{(1)}\ell_3^{(1)})\Delta\widehat{\mathcal{N}}_3^{1,(1)} & -\sinh(\rho_2^{(1)}\ell_3^{(1)})\Delta\widehat{\mathcal{N}}_3^{2,(1)} & -\sinh(\rho_3^{(1)}\ell_3^{(1)})\Delta\widehat{\mathcal{N}}_3^{3,(1)} \\ -\sinh(\rho_1^{(1)}\ell_3^{(1)}) & -\sinh(\rho_2^{(1)}\ell_3^{(1)}) & -\sinh(\rho_3^{(1)}\ell_3^{(1)}) \end{bmatrix},$$

and

$$\mathbf{F}_s^{(1)} \equiv \begin{bmatrix} \sinh(\rho_1^{(1)}\ell_3^{(1)})\Delta\widehat{\mathcal{N}}_2^{1,(1)} & \sinh(\rho_2^{(1)}\ell_3^{(1)})\Delta\widehat{\mathcal{N}}_2^{2,(1)} & \sinh(\rho_3^{(1)}\ell_3^{(1)})\Delta\widehat{\mathcal{N}}_2^{3,(1)} \\ -\cosh(\rho_1^{(1)}\ell_3^{(1)})\Delta\widehat{\mathcal{N}}_3^{1,(1)} & -\cosh(\rho_2^{(1)}\ell_3^{(1)})\Delta\widehat{\mathcal{N}}_3^{2,(1)} & -\cosh(\rho_3^{(1)}\ell_3^{(1)})\Delta\widehat{\mathcal{N}}_3^{3,(1)} \\ -\cosh(\rho_1^{(1)}\ell_3^{(1)}) & -\cosh(\rho_2^{(1)}\ell_3^{(1)}) & -\cosh(\rho_3^{(1)}\ell_3^{(1)}) \end{bmatrix}.$$

Finally, from (14), (15), (19) and (20) we obtain the set of 12 homogeneous algebraic equations, which is put in the compact form

$$\underbrace{\begin{bmatrix} -\mathbf{H}_c^{(1)} & \mathbf{0} & \mathbf{H}_c^{(2)} & \mathbf{0} \\ \mathbf{0} & -\mathbf{H}_s^{(1)} & \mathbf{0} & \mathbf{H}_s^{(2)} \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_c^{(2)} & -\mathbf{F}_s^{(2)} \\ \mathbf{F}_c^{(1)} & -\mathbf{F}_s^{(1)} & \mathbf{0} & \mathbf{0} \end{bmatrix}}_{\mathbf{M}(\omega_1,\omega_2,d_0)} \underbrace{\begin{bmatrix} \boldsymbol{\xi}^{c,(1)} \\ \boldsymbol{\xi}^{s,(1)} \\ \boldsymbol{\xi}^{c,(2)} \\ \boldsymbol{\xi}^{s,(2)} \end{bmatrix}}_{\boldsymbol{\Xi}} = \mathbf{0}. \tag{21}$$

In this expression, $\mathbf{M}$ is the matrix defined in equation (A5.13) in the main text.