Magneto-Rheological Elastomers (MREs): from micro-deformation mechanisms to macroscopic instabilities and applications

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European Research Council
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Active materials in engineering applications

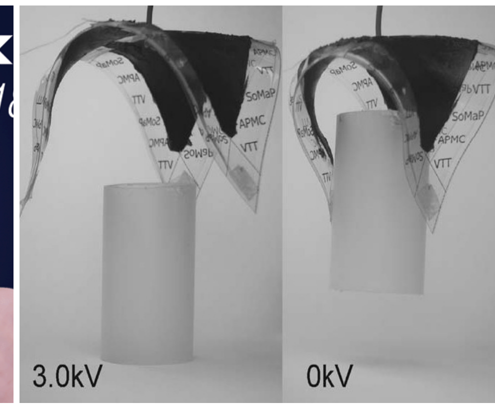
Energy Harvesters

hard

soft

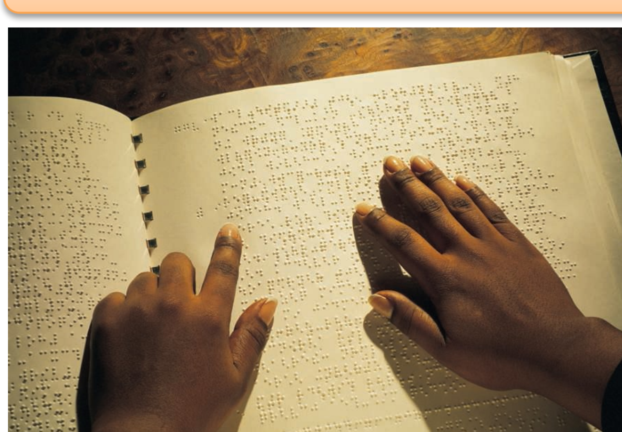
Actuators

hard

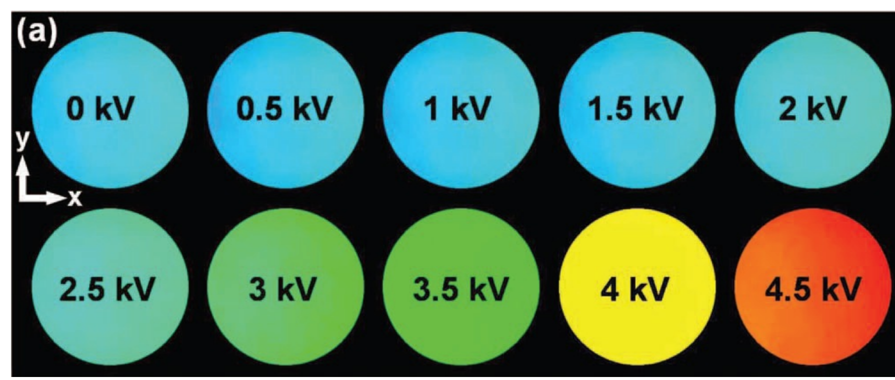


soft

Refreshable Braille displays



Tunable color displays



SRI (2009); Kofod et al. (2007); Carpi et al. (2010); Aschwanden & Stemmer (2006)

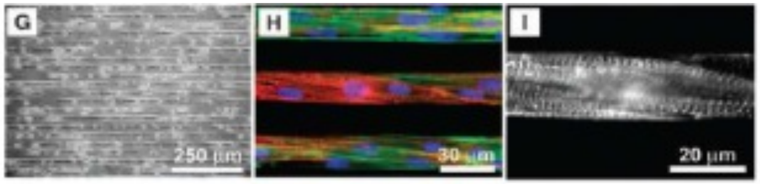
Biological and bio-inspired active materials

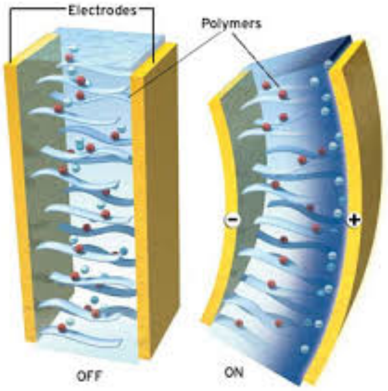
Human Heart

Human muscles

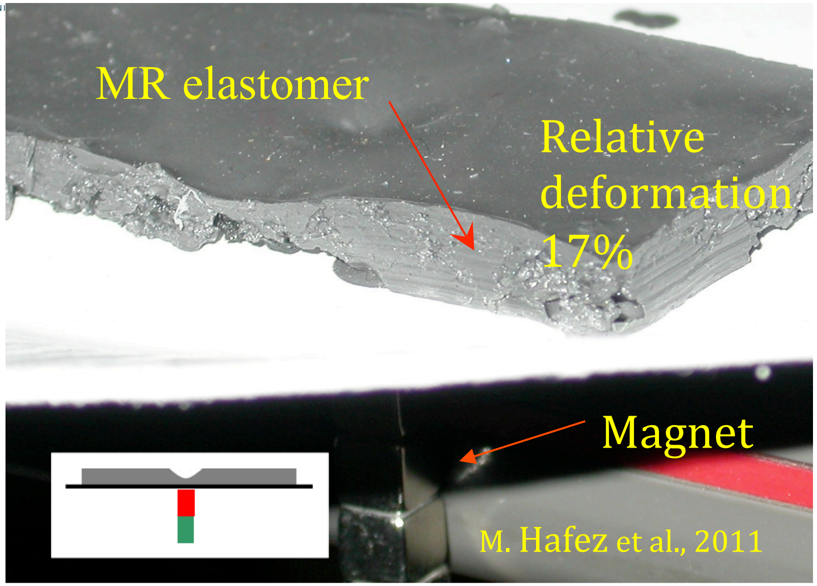
- Driven by electro-chemical actuation signals to create mechanical deformation

- Polymer thin films

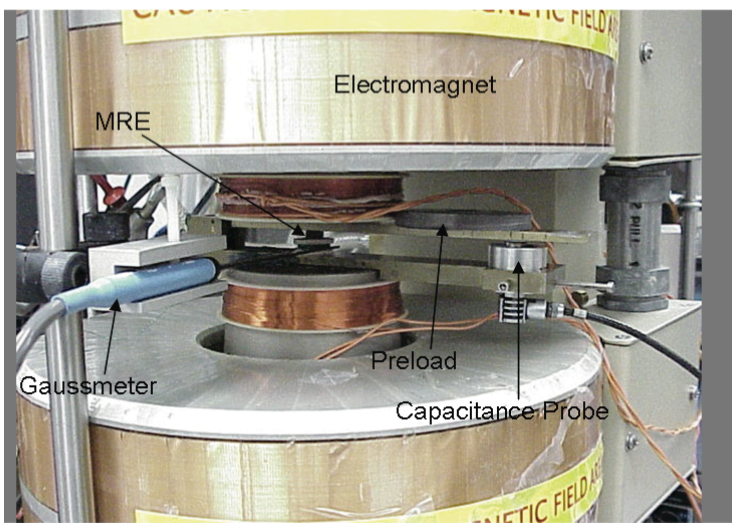


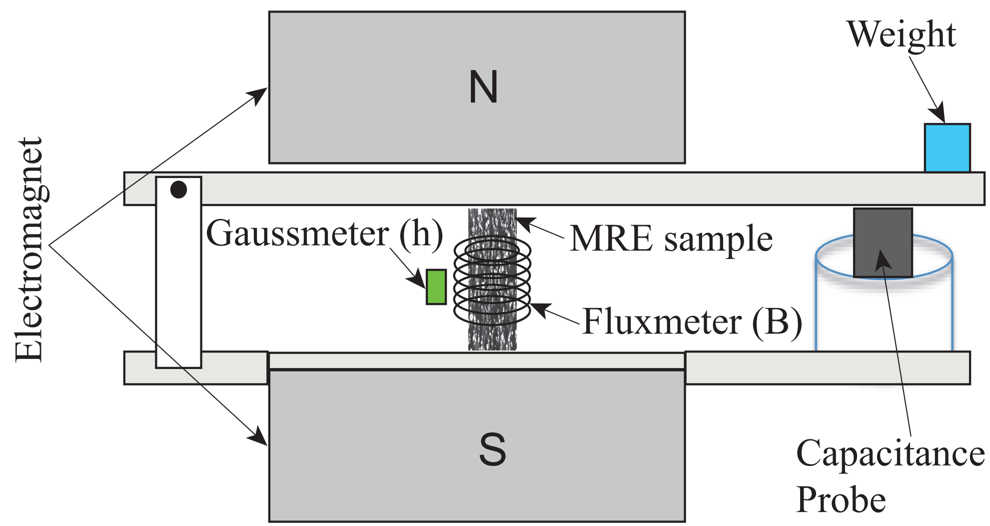


Magneto-active materials



- Torsional suspension mounts (change stiffness upon application of electric current)
- Vibration dampers, Haptics

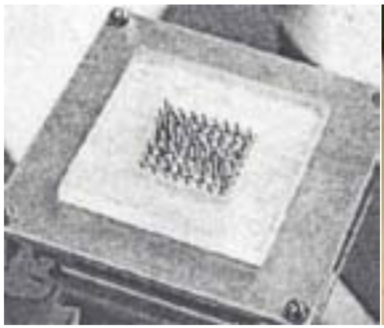




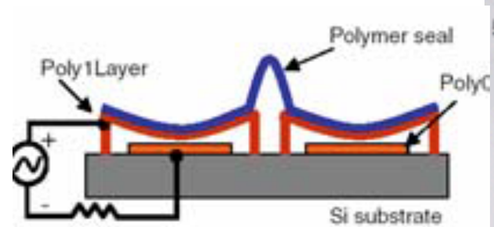
[Ginder et al., 1999, Danas et al. 2012]

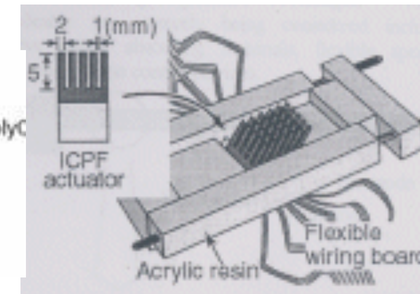
Active materials in haptic interfaces

Tactile Teleoperation : Vibration

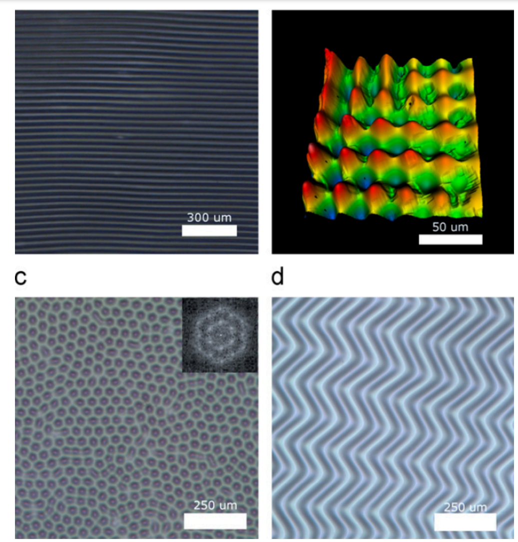


Haptic Interface : EAPs





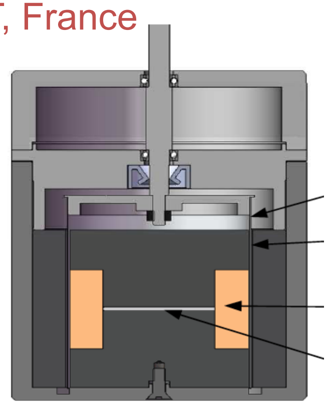
Thin-metal films on PDMS



Audoly & Boudaoud, 2009; Breid and Croscby, 2009; Cai et al. JMPS, 2011

MRF 1-dof haptics: in-Vehicle

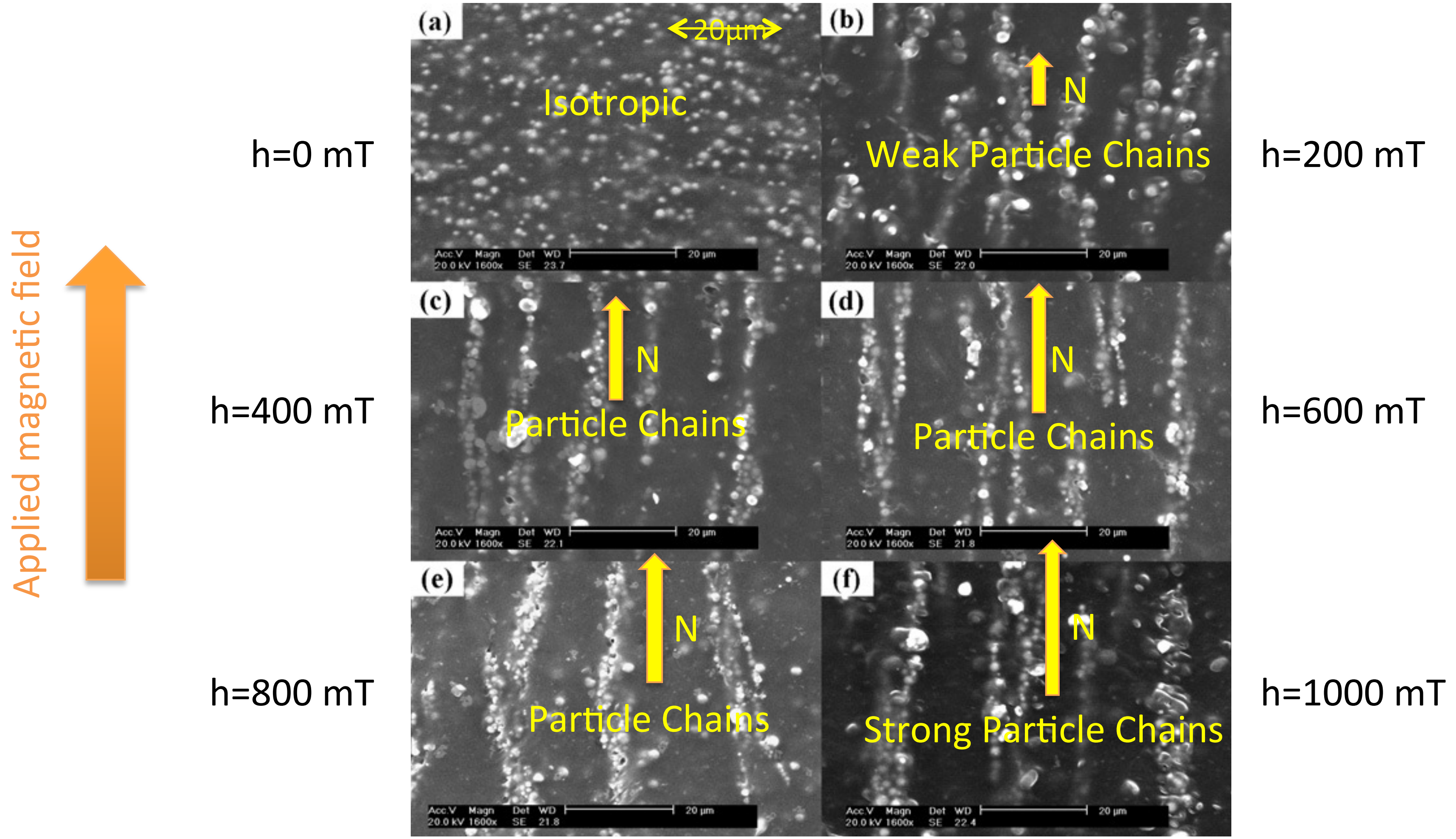
Design made by CEA LIST, France



Periquet and Lozada, 2012

Particle chain microstructures

SEM Images of 6 MREs with 11% ferromagnetic particles



T : Tesla

[Chen et al., Smart Mater. Struct, 2007]

Uniform vs non-Uniform magnetic fields

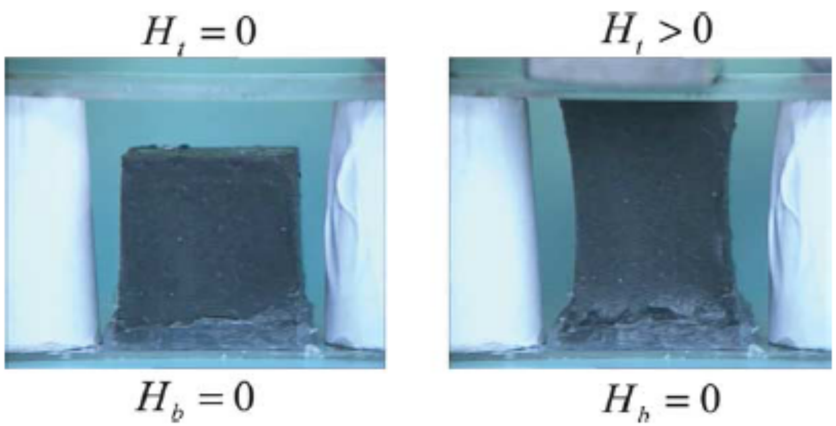


Fig. 1. Non-contact mode of elongation of a magnetite loaded PDMS network due to non-uniform magnetic field H_t and H_b represents the field intensity below (H_b) and above (H_t) the magnetoelast.

Field-Particle interactions dominate when applied field is non-uniform.

Particle-Particle interactions dominate when applied field is uniform.

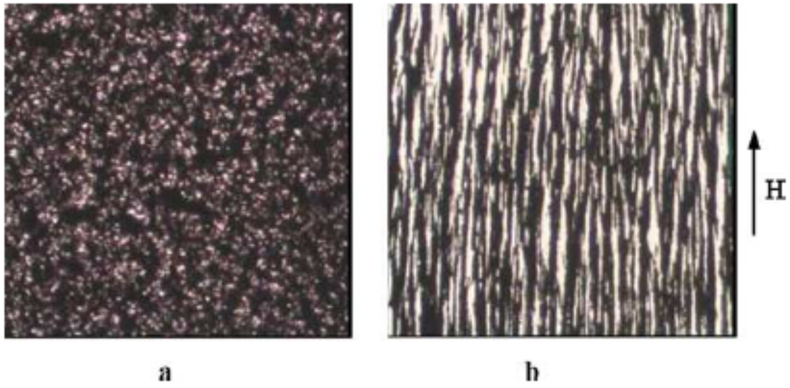


Fig. 2. Formation of bundles of magnetite particles in silicon oil, parallel to the field direction as seen by microscope. The concentration of magnetite in the mixture is 5 wt%. (a) No external magnetic field. (b) The magnetic induction 50 mT.

[Varga et al., Polymer, 2007]

Plan of the talk

- Fabrication and Experiments
  - Particle-chain manufacturing
  - Uniaxial experiments under magneto-mechanical loads.
  - A micro-deformation mechanism of inter-particle motion is proposed.

- Finite element formulations
  - A unique minimum variational formulation.
  - Simulations of multi-particle systems.
  - Periodic unit-cells.

- Hierarchical Marginally stable MREs
  - A simple film/substrate structure.
  - Asymptotic analysis and full analytical solutions.
  - FE simulations and preliminary experiments.

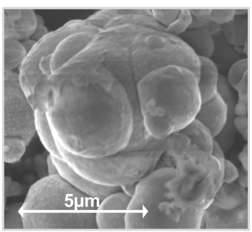
FABRICATION

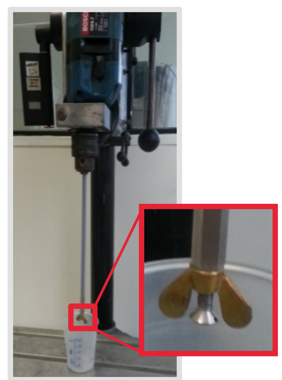
Pössinger et al., SPIE Microtechnologies (2013)

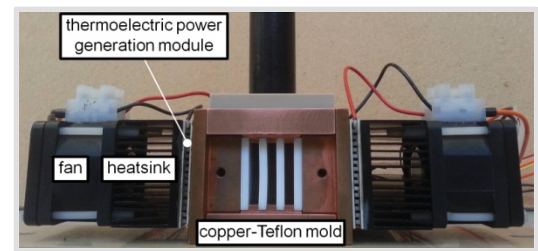
Fabrication procedure

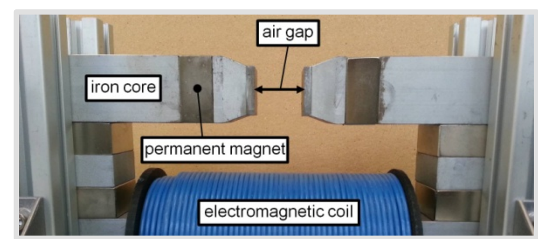
- **Particles**
  - Fe powder
  - Average diameter size 3.5µm

- **Matrix**
  - Soft silicone, Shore 00-20, E=0.03Mpa
  - Two-part liquid RTV system





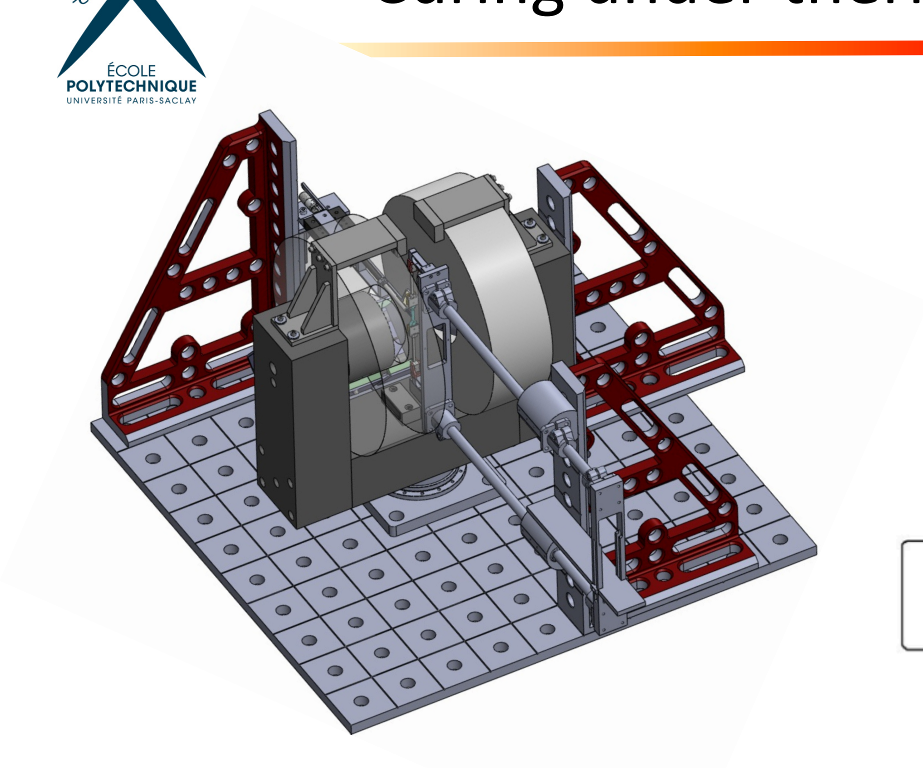


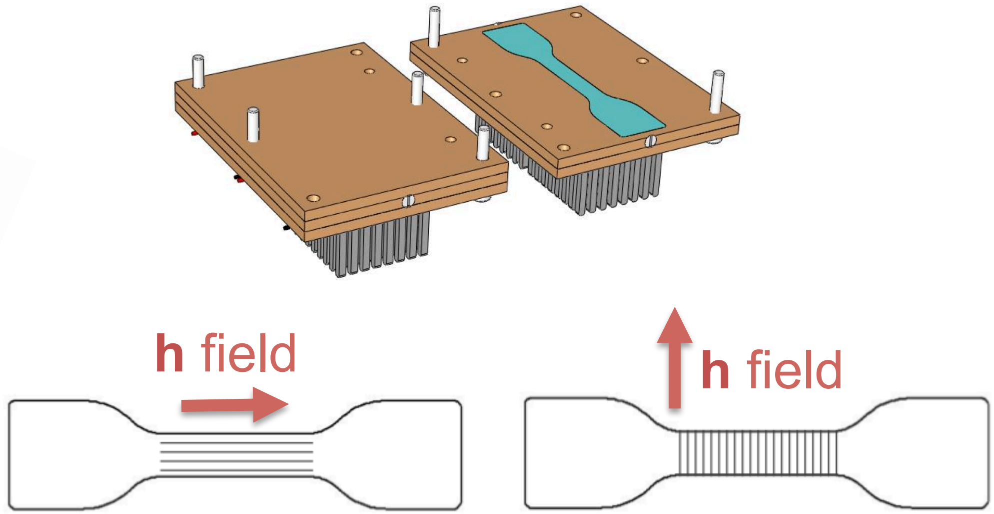


Pretreatment → Weighing → Mixing → Degassing → Molding → Curing

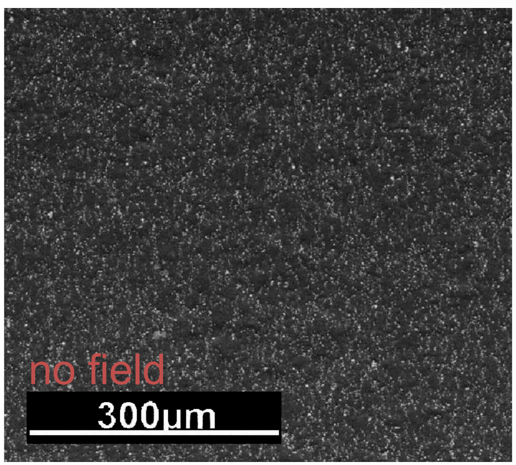
Pot life (spanning Mixing to Molding)

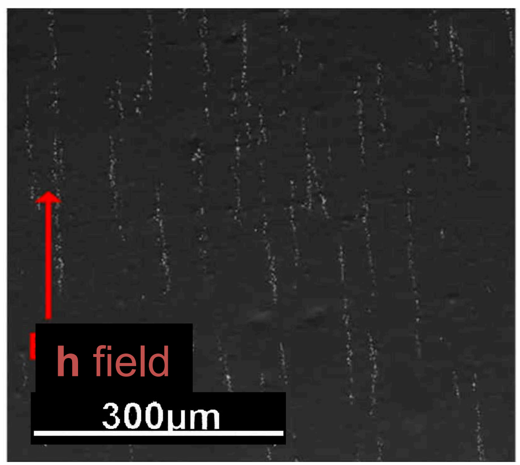
Curing under thermal and magnetic fields

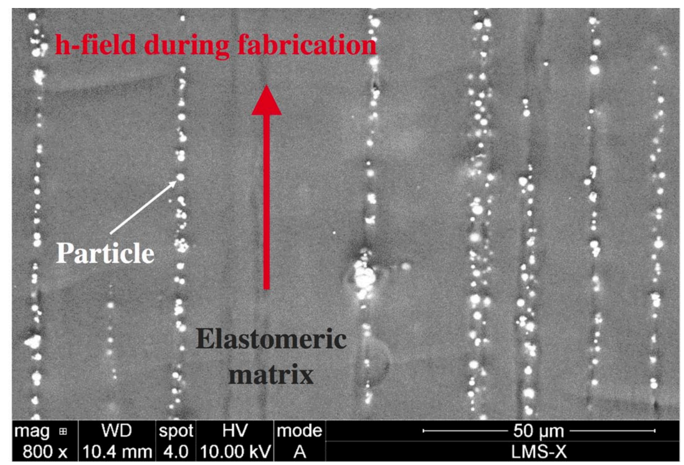




Alignement of particles in field direction



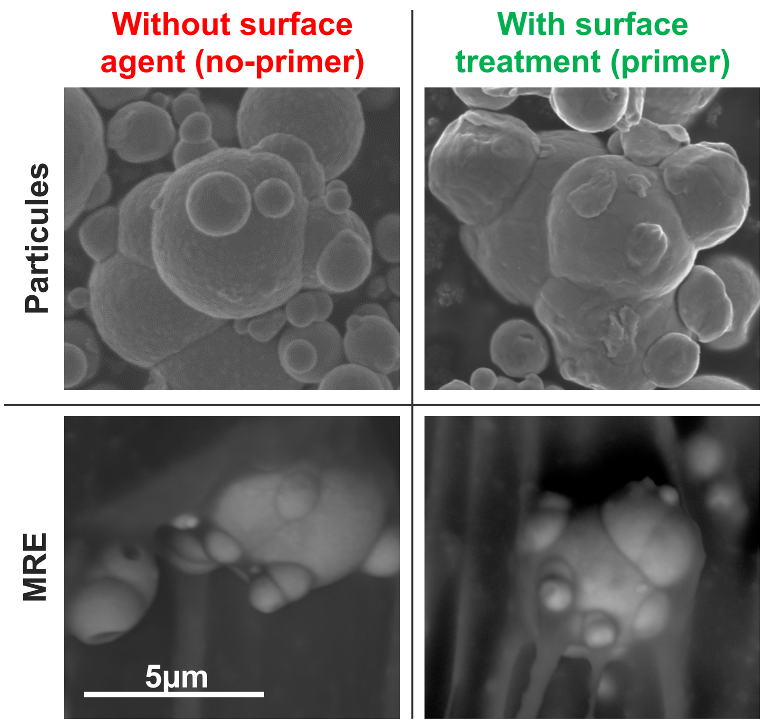




[Possinger et al., 2014]

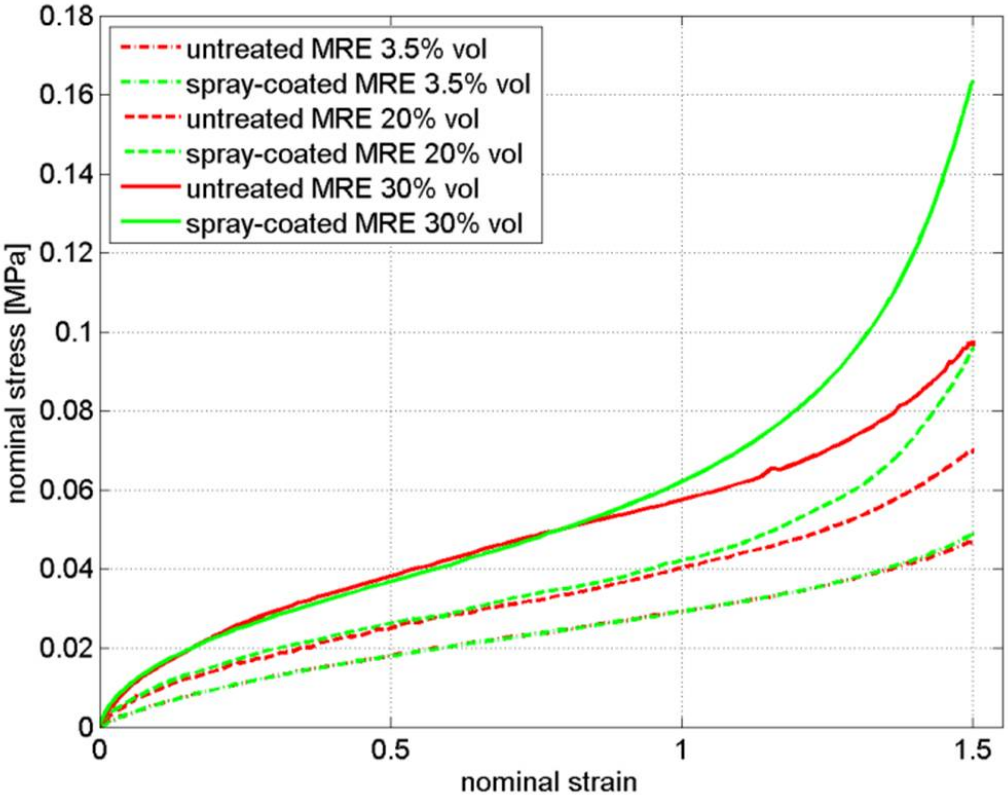
Effect of surface treatment

Isotropic distributions of particles



Images taken at the SEM of LMS

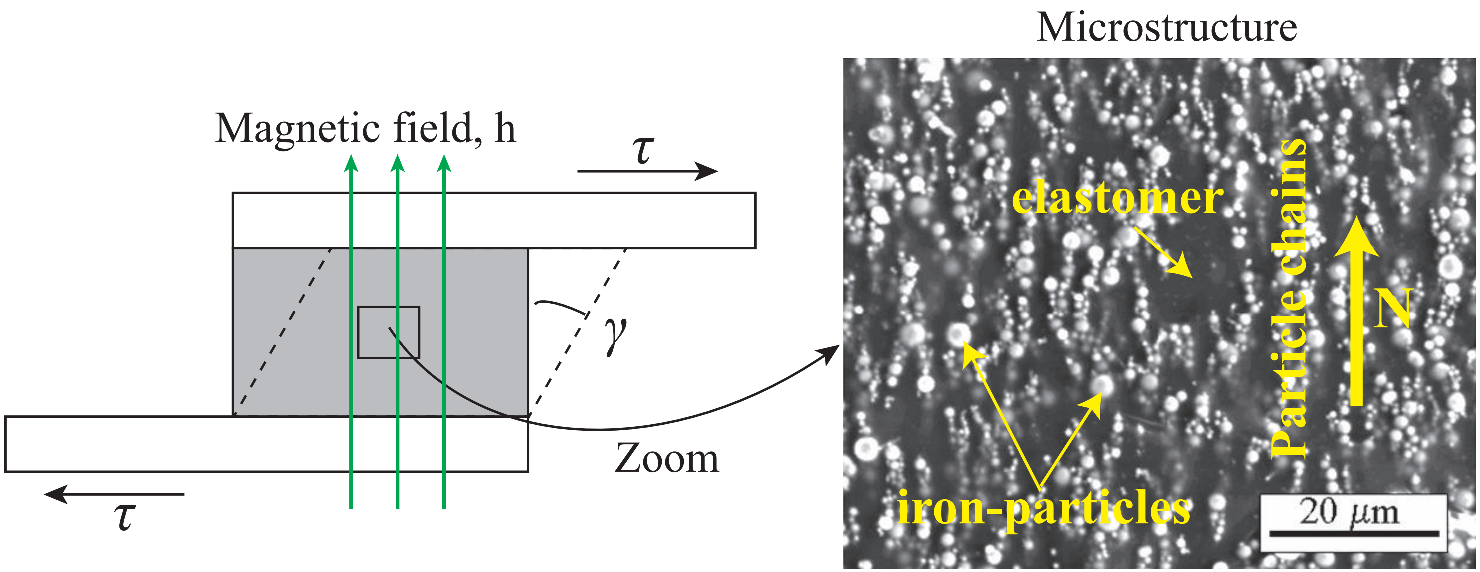
Uniaxial tension at different vol% along the particle chains



EXPERIMENTS

Danas, Kankanala & Triantafyllidis, JMPS, (2012)

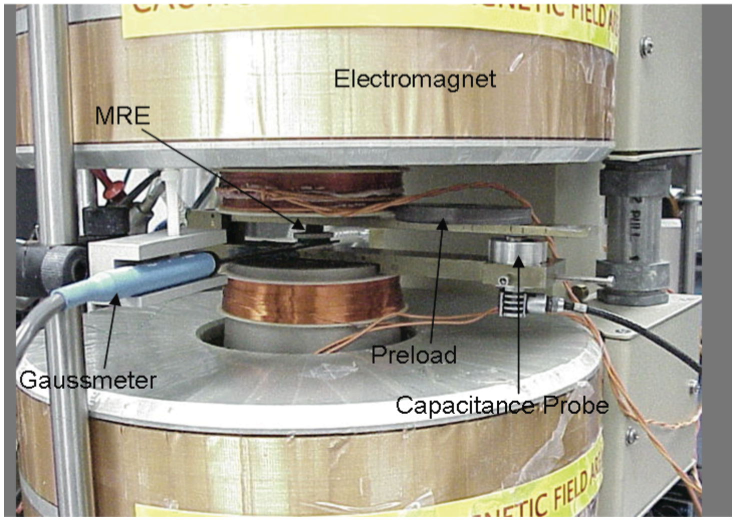
Micrographs & Material Fabrication

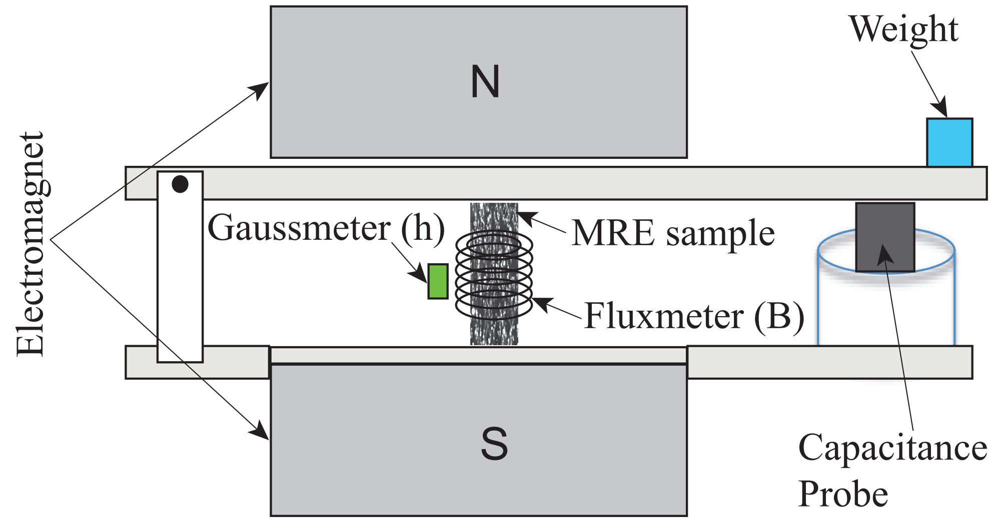


[Ginder et al., 1999]

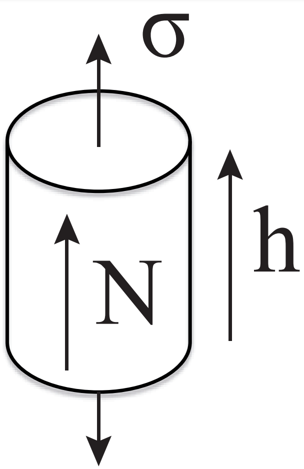
- MRE comprises 25% iron particles of 0.5-5μm size.
- MRE cured in 0.5T magnetic field.
- Curing process leads to formation of particle chains.

Setup for magnetostriction

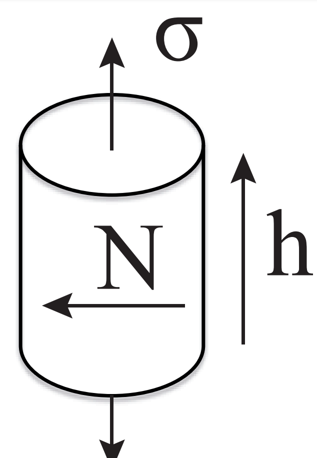




- Particle Chains parallel to magnetic field



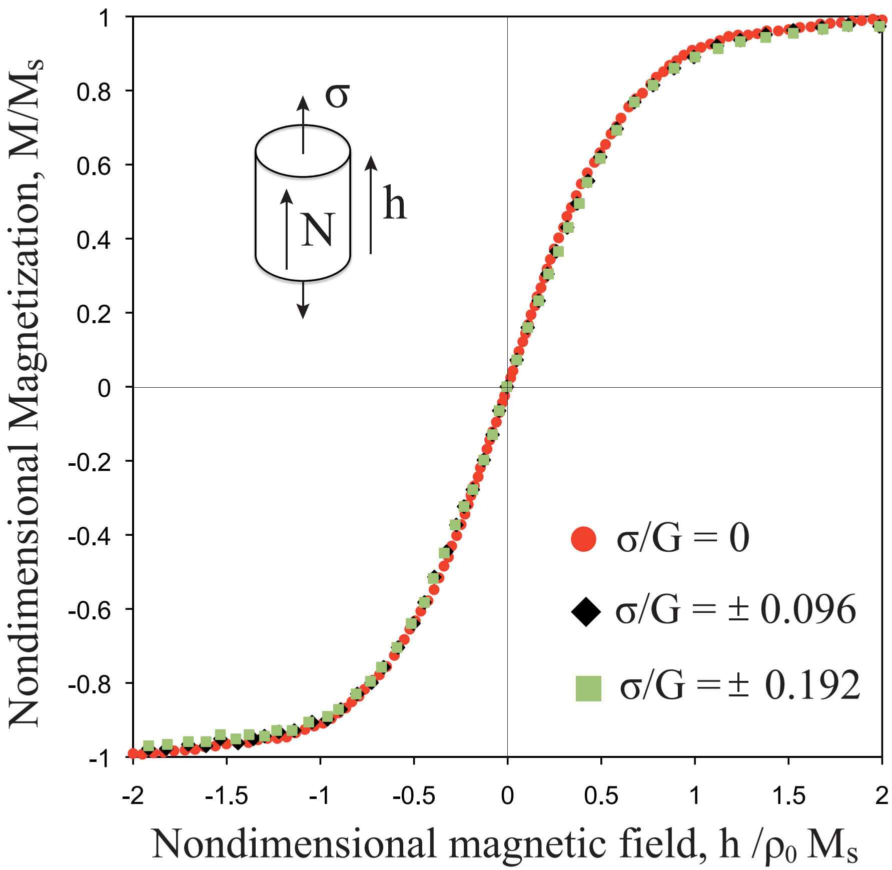
- Particle Chains perpendicular to magnetic field

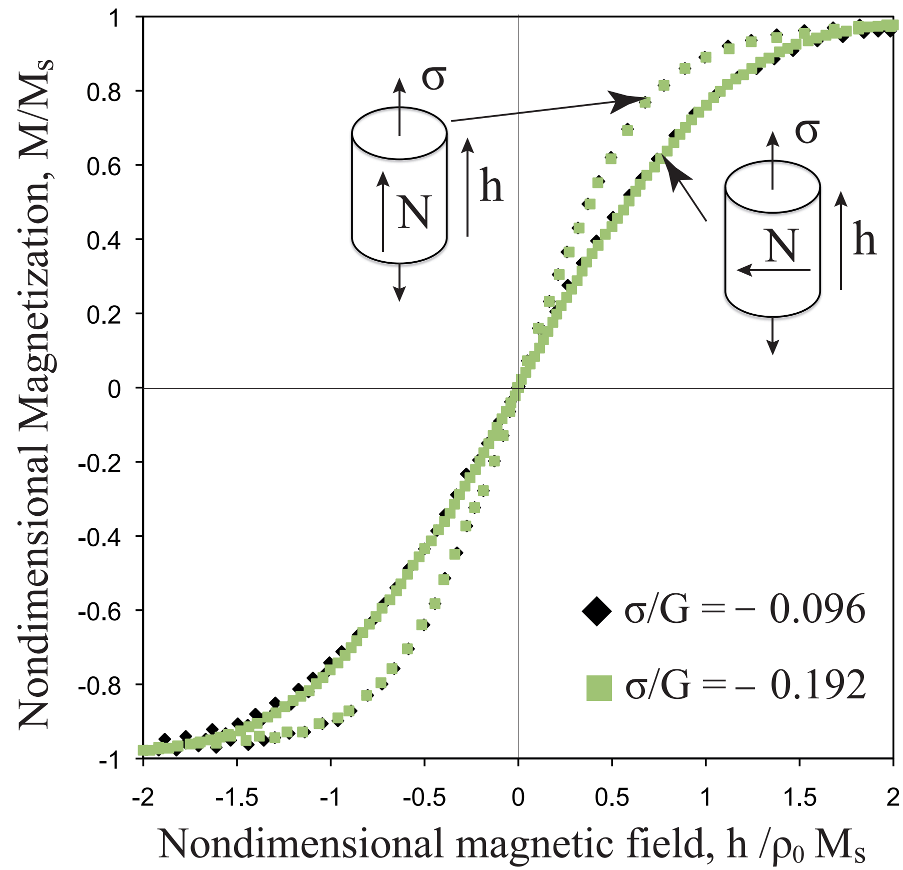


Magnetization – Experiments

Effect of preload on magnetization response

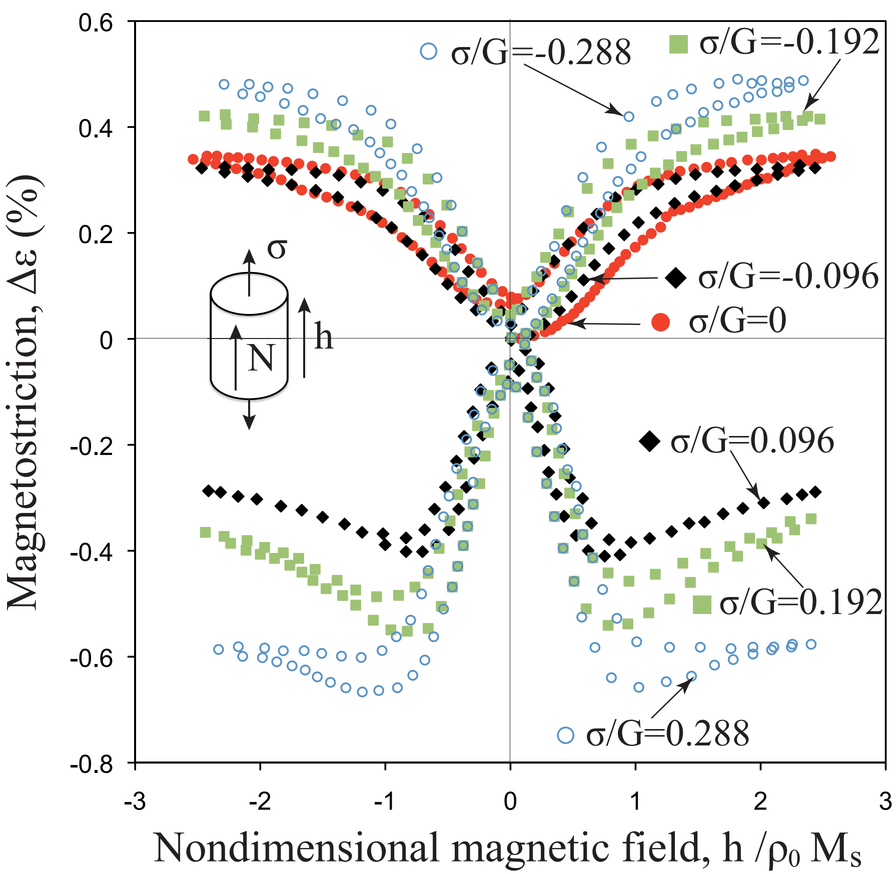
Effect of particle chain orientation on magnetization response



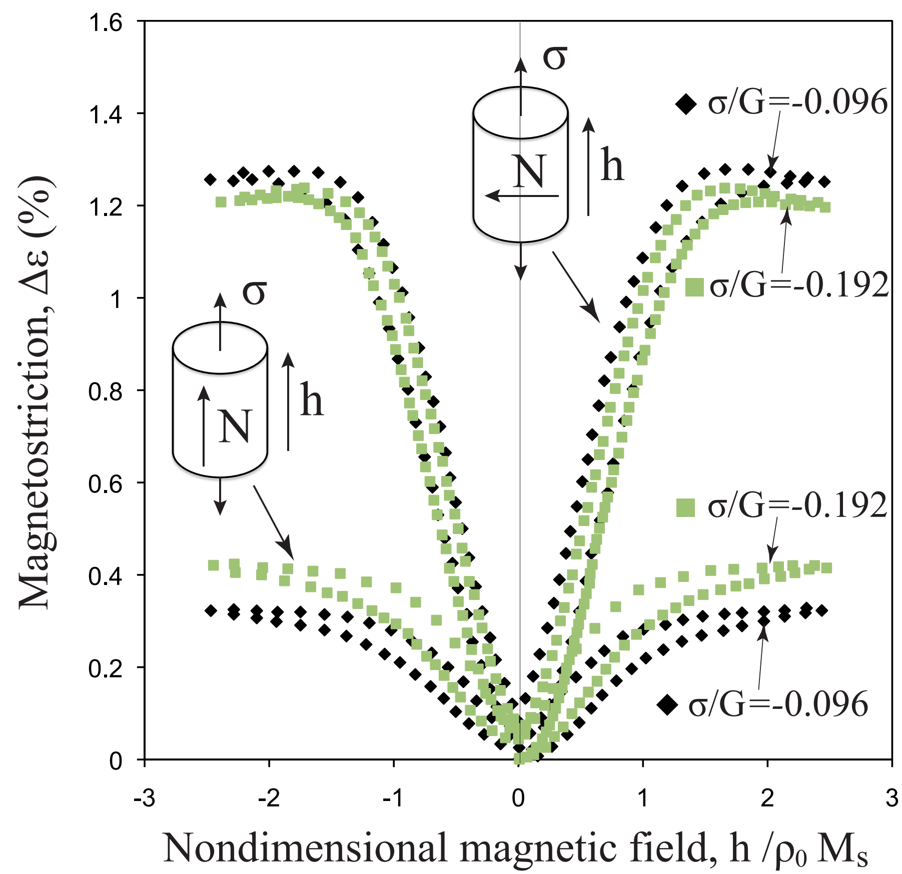


Magnetostriction – Experiments

Effect of preload on magnetization response

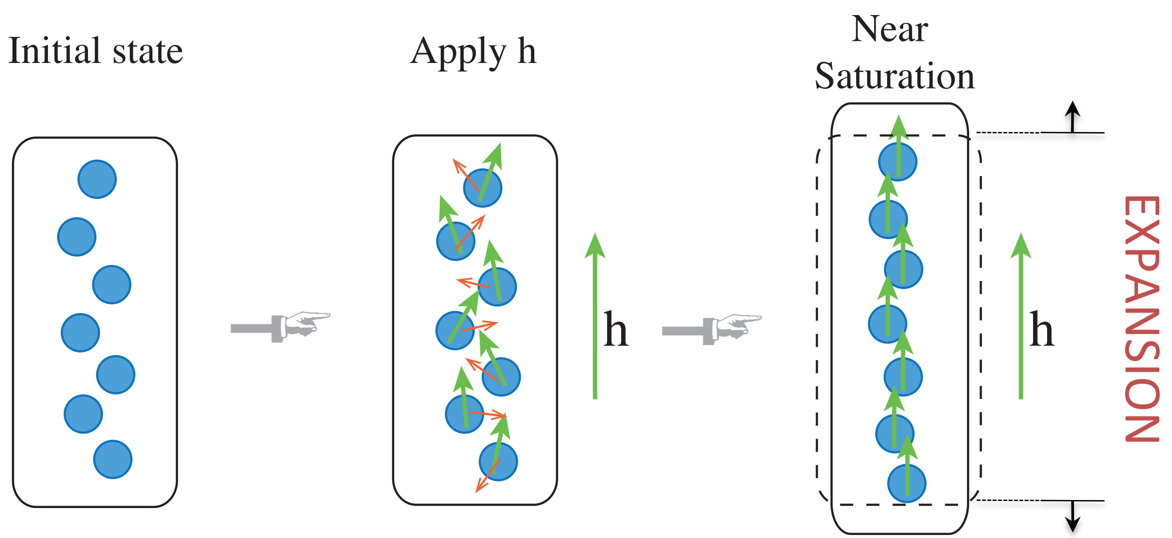


Effect of particle chain orientation on magnetization response

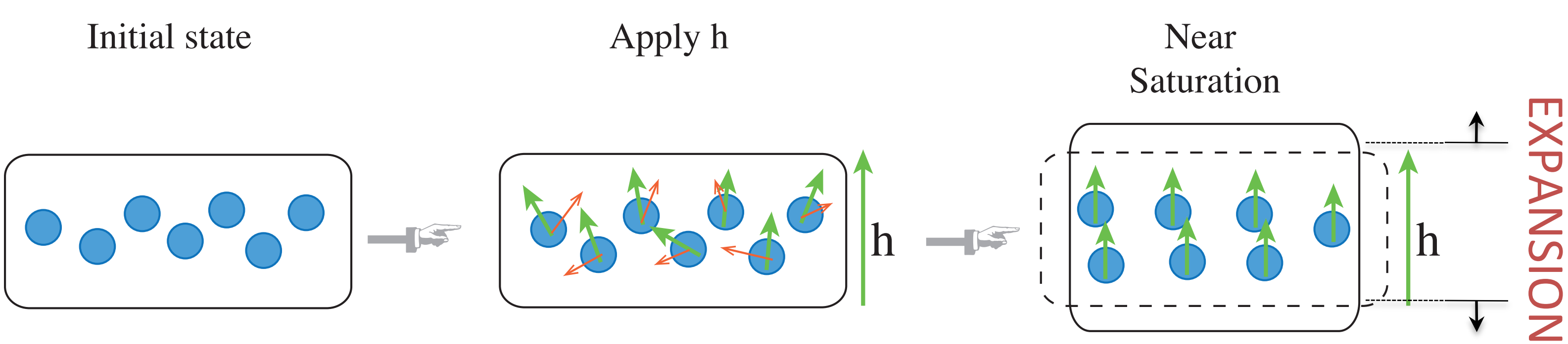


Microdeformation mechanism – Conjecture

Parallel configuration, $\sigma/G \leq 0$



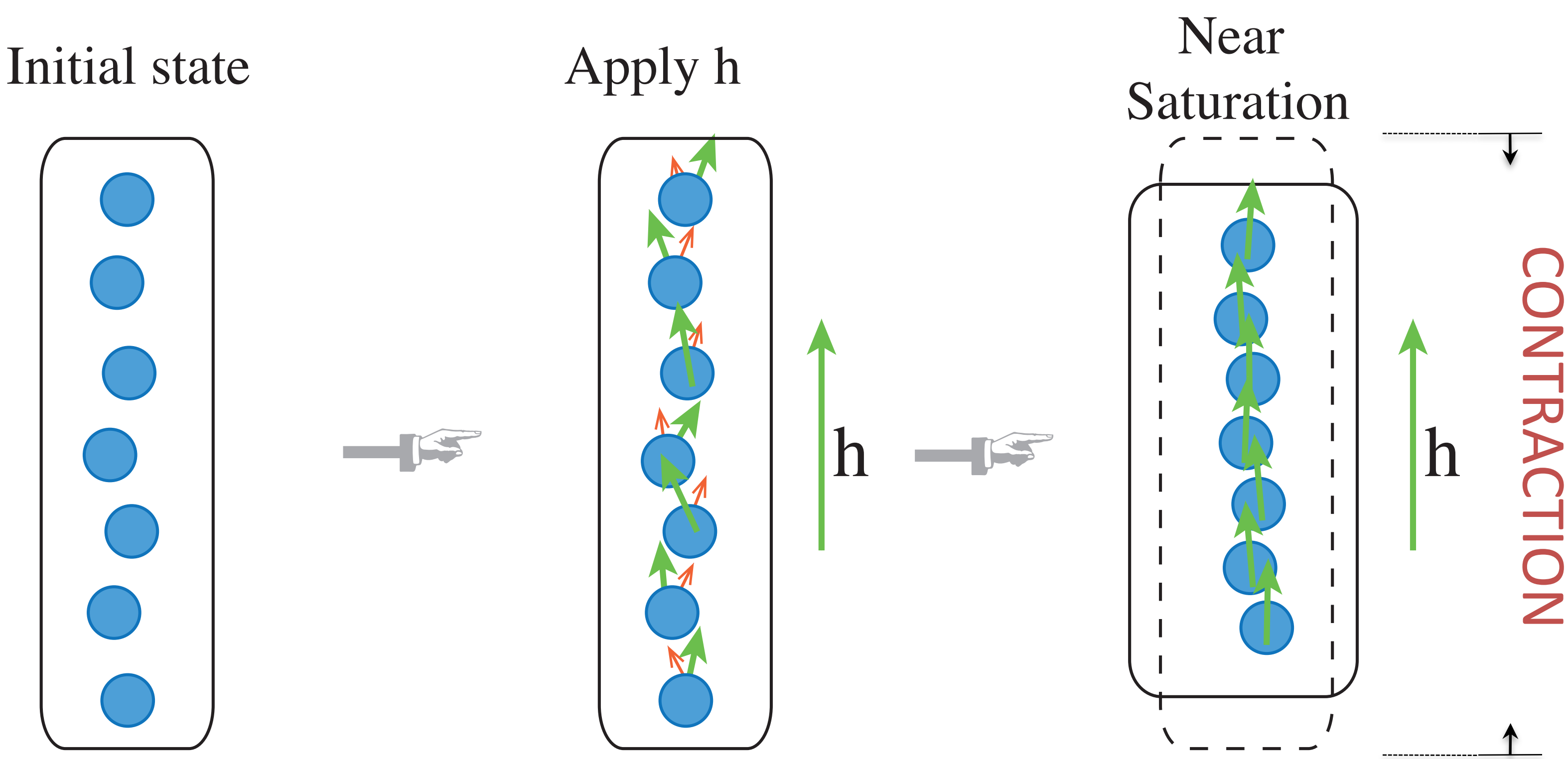
Perpendicular configuration, $\sigma/G < 0$



[Klingenberg & Zukoski, 1990; Bossis & Lemaire, 1991]

Microdeformation mechanism – Conjecture

Parallel configuration, $\sigma/G \geq 0.075$



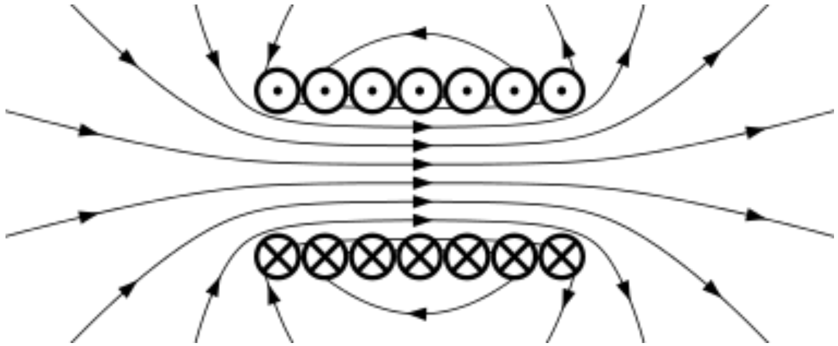
[Klingenberg & Zukoski, 1990; Bossis & Lemaire, 1991]

Finite Element Implementations

Work in progress

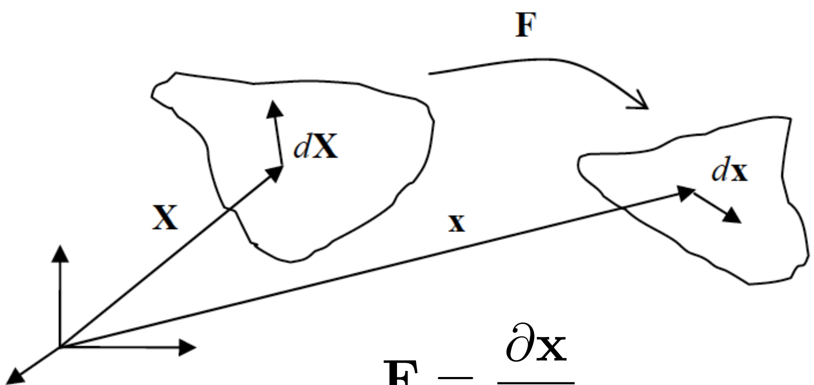
The fields : some basic notions

- Magnetic field, **b**



- Displacement **u**

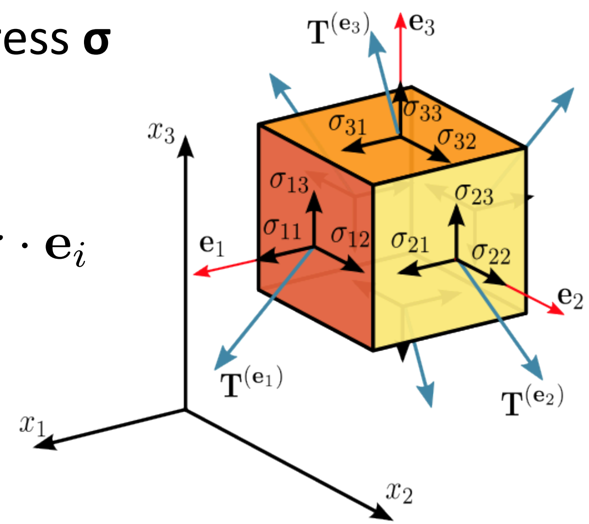
- Deformation **F**



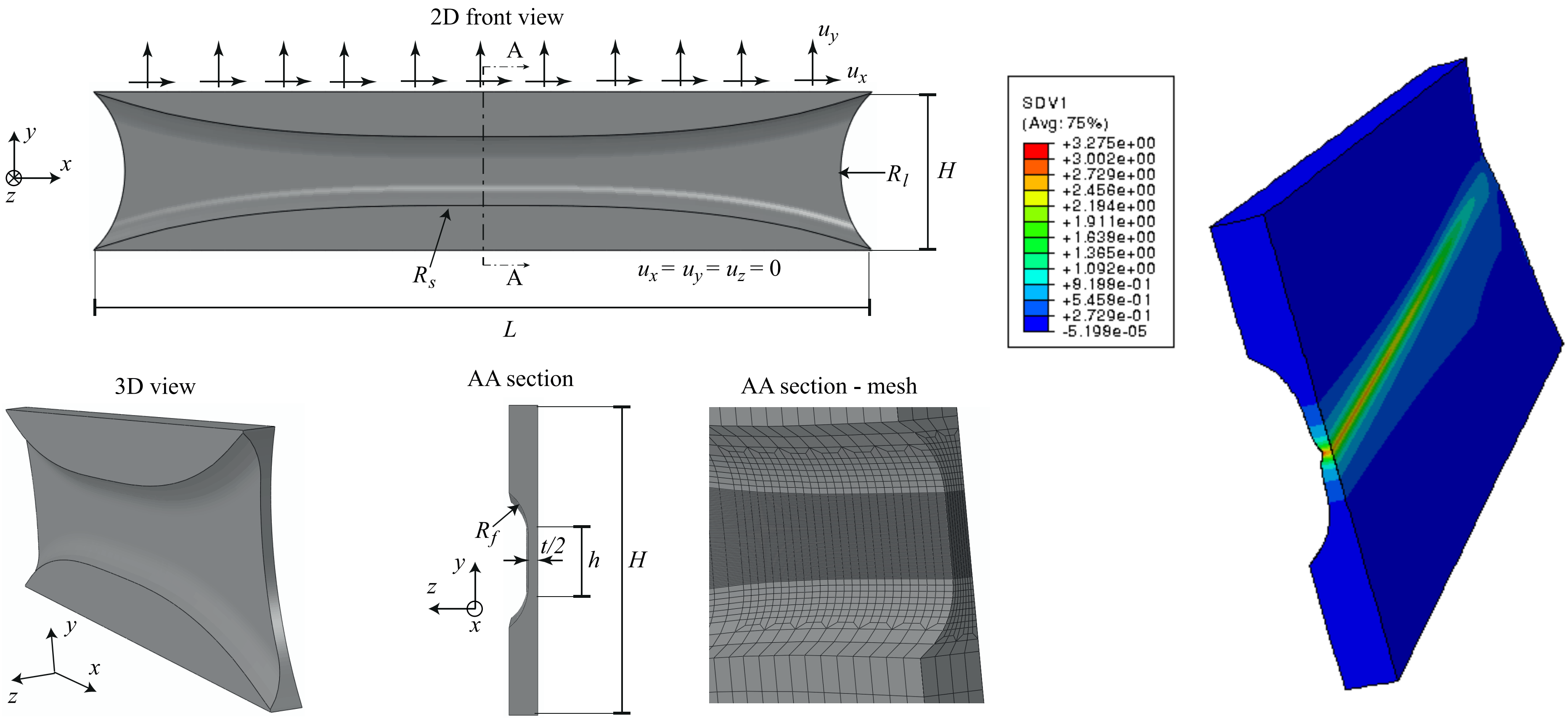
$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$$

- Stress **σ**

$$\mathbf{T}^{e_i} = \boldsymbol{\sigma} \cdot \mathbf{e}_i$$



We do this since long time in engineering



- 3D geometry
- Strain map

[Danas & Aravas, 2012]

Potential Energy in Reference Configuration

- Potential energy associated with direct formulation:

$$\mathcal{P}(\mathbf{u},\mathbf{M},\alpha) = \int_V \rho_0 \left[\Psi\left(\mathbf{C},\mathbf{M}\right) - \mu_0\,\mathbf{M}\cdot\widehat{\mathbf{h}} - \mathbf{f}\cdot\mathbf{u}\right]\mathrm{d}V - \int_{\partial V}\mathbf{T}\cdot\mathbf{u}\,\mathrm{d}A$$
$$+ \int_{\mathbb{R}^3}\frac{1}{2\mu_0\,J}\left|\mathbf{F}\cdot(\boldsymbol{\nabla}\times\boldsymbol{\alpha}) - \mu_0\,\rho_0\,\mathbf{M}\right|^2\mathrm{d}V$$

NOTATION

$$\widetilde{\mathbf{B}} = \boldsymbol{\nabla}\times\boldsymbol{\alpha}$$
$$\mathbf{B} = J\,\mathbf{F}^{-1}\cdot\mathbf{b}$$
$$\mathbf{B} = \widehat{\mathbf{B}} + \widetilde{\mathbf{B}}$$
$$\mathbf{b} = \mu_0\left(\mathbf{h} + \rho\,\mathbf{M}\right)$$

$$\rho_0 = \rho\,J\ :$$
$$\boldsymbol{\nabla} = \partial/\partial\mathbf{X}$$
$$J = \det\mathbf{F}\ :$$
$$\mathbf{F} = \mathbf{I} + \mathbf{u}\boldsymbol{\nabla}$$

- Variation of P w.r.t the independent variables gives field equations:

$$\delta\mathcal{P}(\mathbf{u},\mathbf{M},\boldsymbol{\alpha}) = \mathcal{P}_{,\mathbf{u}}\,\delta\mathbf{u} + \mathcal{P}_{,\mathbf{M}}\,\delta\mathbf{M} + \mathcal{P}_{,\boldsymbol{\alpha}}\,\delta\boldsymbol{\alpha} = 0$$

Summary of Field Equations (current config)

	Governing equations :	*Boundary/Interface conditions :*
Amper's law :	$\nabla \times \mathbf{h} = \mathbf{0}$	$\mathbf{n} \times [\![\mathbf{h}]\!] = 0$
Zero net flux law:	$\nabla \cdot \mathbf{b} = \mathbf{0}$	$\mathbf{n} \cdot [\![\mathbf{b}]\!] = 0$
Linear momentum:	$\nabla \cdot \sigma + \rho\, \mathbf{f} = \mathbf{0}$	$\mathbf{n} \cdot [\![\sigma]\!] = \mathbf{t}$

Constitutive equations in current configuration

- Total Stress (using objectivity of Ψ) :

$$\boldsymbol{\sigma} = \rho \left[2\mathbf{F} \cdot \frac{\partial \Psi}{\partial \mathbf{C}} \cdot \mathbf{F}^T + \mu_0 \left(\mathbf{M}\mathbf{h} + \mathbf{h}\mathbf{M} \right) \right] + \boxed{\mu_0 \left[\mathbf{h}\mathbf{h} - \frac{1}{2} (\mathbf{h} \cdot \mathbf{h}) \mathbf{I} \right]} = \boldsymbol{\sigma}^T$$

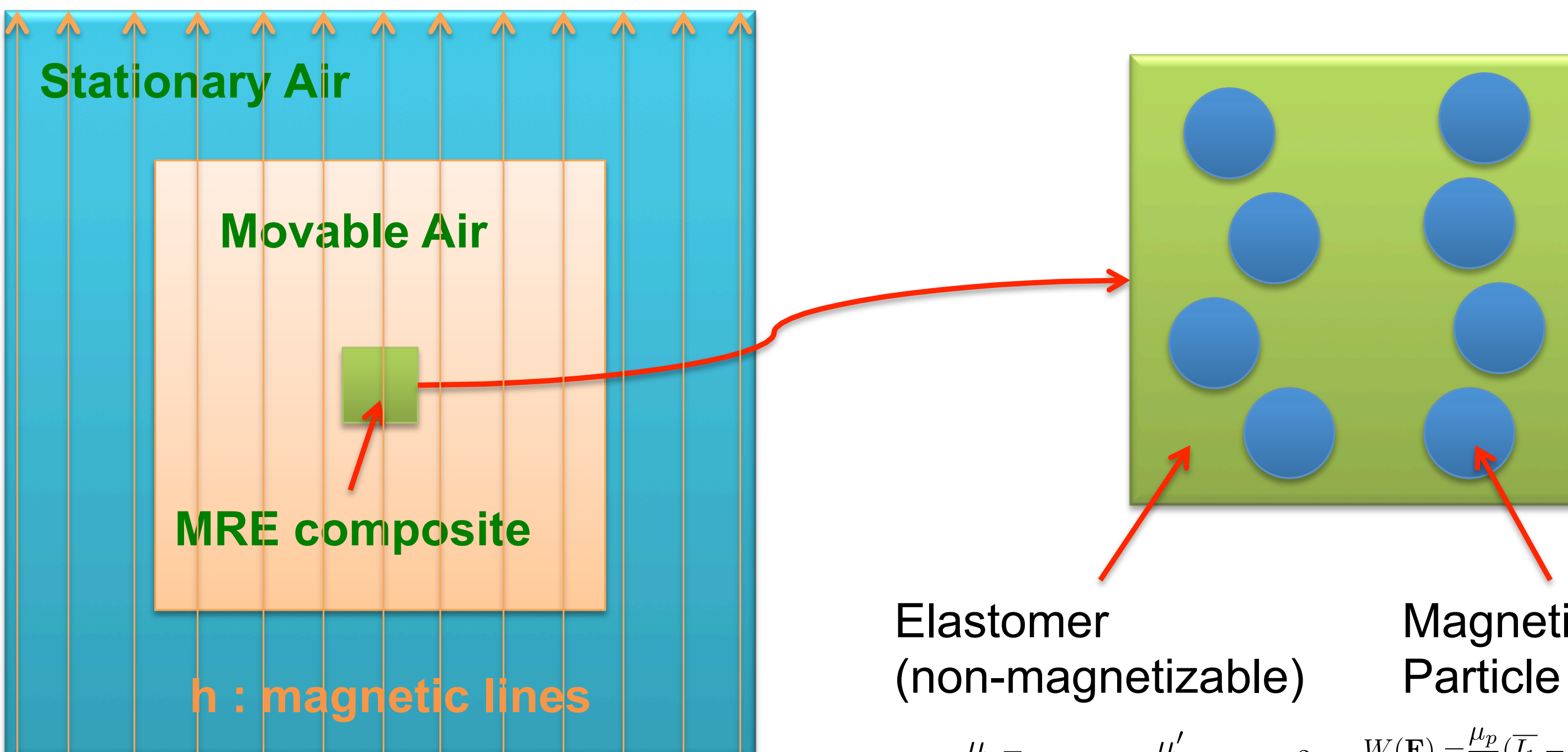
- Magnetic relations : $\mu_0\, \mathbf{h} = \dfrac{\partial \Psi}{\partial \mathbf{M}}$ $\quad \mathbf{b} = \mu_0 \left(\mathbf{h} + \rho\, \mathbf{M} \right)$

$\mathbf{b}$: magnetic field	$\mathbf{h}$: *h*-field	$\mathbf{M}$: spec. magnetization
ρ : mat. density	$\mathbf{F}$: deformation gradient	$\mathbf{f}$: mech. body force

Boundary Value Problems including air

Entire 2D Geometry including both air & MRE composite

2D Geometry of a representative MRE composite



Elastomer (non-magnetizable)

$$W(\mathbf{F}) = \frac{\mu}{2}(\overline{I}_1 - 3) + \frac{\mu'}{2}(J-1)^2$$

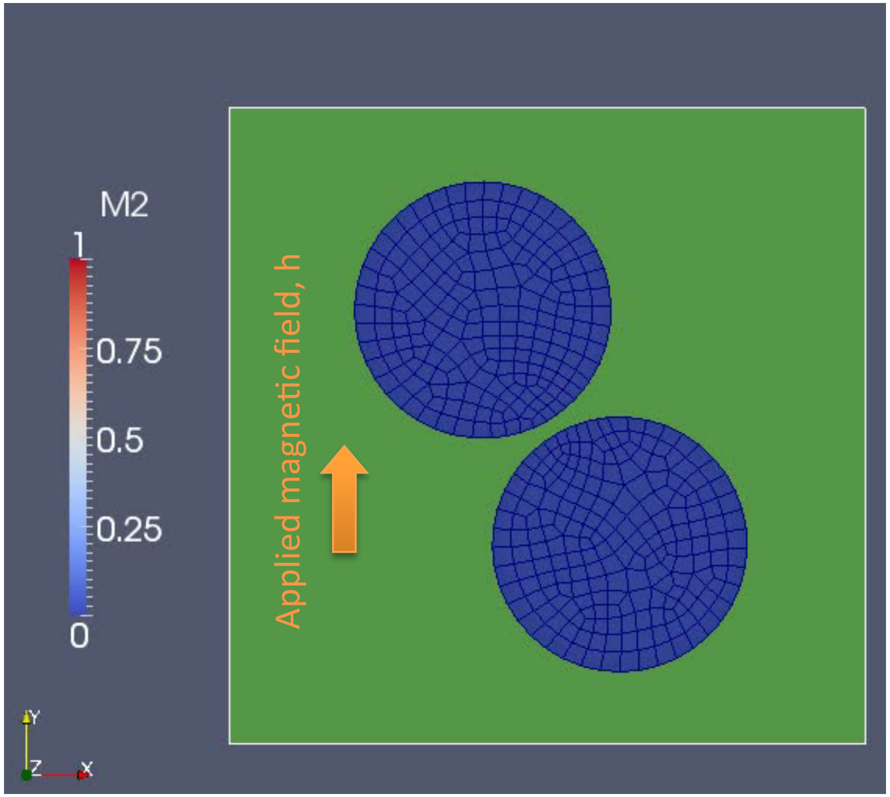
Magnetizable Particle (e.g., Fe)

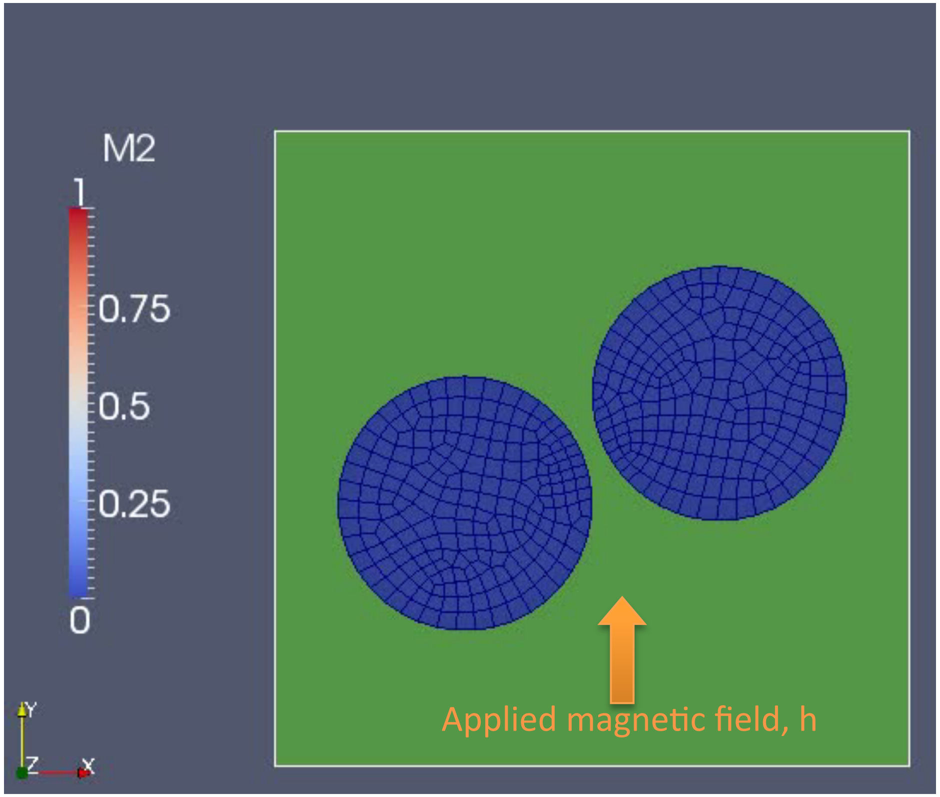
$$W(\mathbf{F}) = \frac{\mu_p}{2}(\overline{I_1} - 3) + \frac{\mu'_p}{2}(J-1)^2 + c_M\,\mathbf{M}\cdot\mathbf{M} + f_{sat}(\mathbf{M}\cdot\mathbf{M})$$

Two-particle FE analysis

2-particle chain parallel to applied magnetic field

2-particle chain perpendicular to applied magnetic field



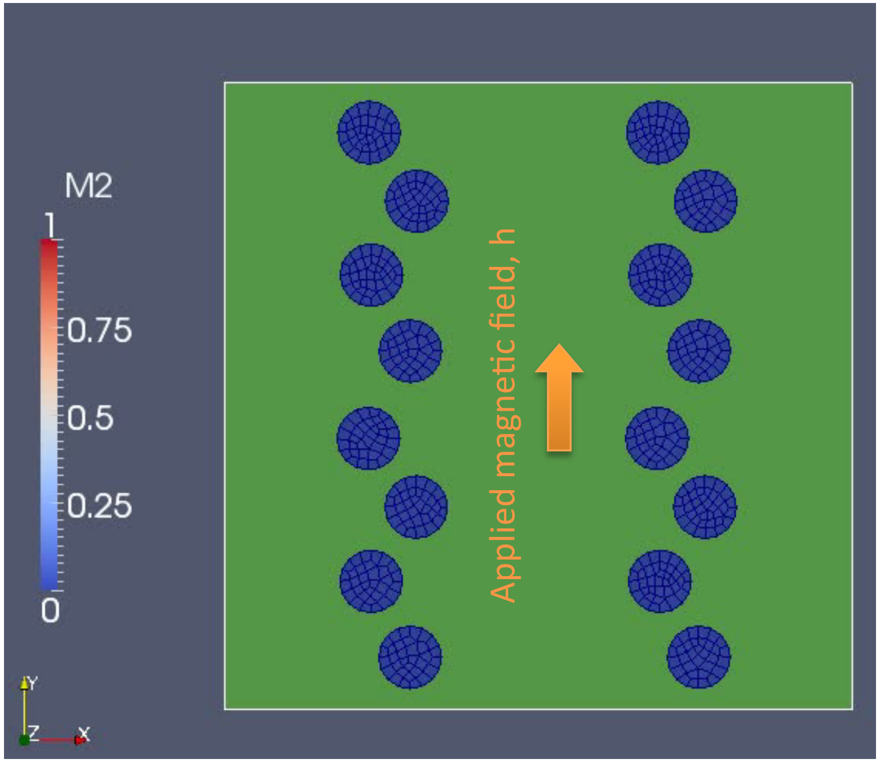


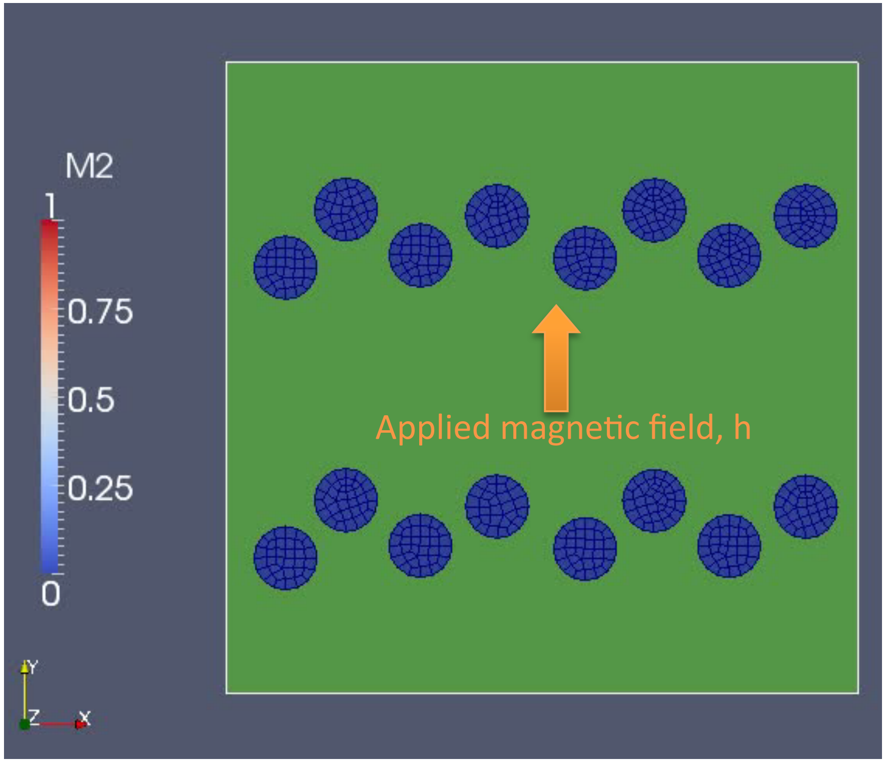
Extension of MRE under applied magnetic field

16-particle FE analysis

Chains parallel to applied magnetic field

Chains perpendicular to applied magnetic field

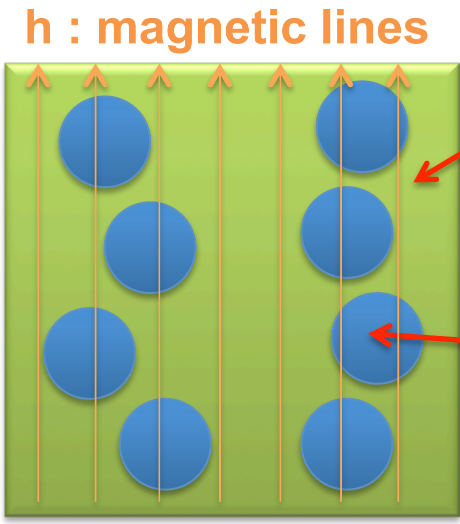




Extension of MRE under applied magnetic field

Two-particle Periodic FE analysis

Representative 2D periodic unit-cell



$\mu_p >> \mu$

- Elastomer (non-magnetizable)

$$W(\mathbf{F}) = \frac{\mu}{2}(\overline{I}_1 - 3) + \frac{\mu'}{2}(J-1)^2$$

- Rigid Magnetizable particle (e.g., Fe)

$$W(\mathbf{F}) = \frac{\mu_p}{2}(\overline{I}_1 - 3) + \frac{\mu_p'}{2}(J-1)^2 + c_M\,\mathbf{M}\cdot\mathbf{M} + f_{sat}(\mathbf{M}\cdot\mathbf{M})$$

Periodic BC's

- Displacement & Stress field

$$u_i = \left(\overline{F}_{ij} - \delta_{ij}\right) X_j + u_i^*$$

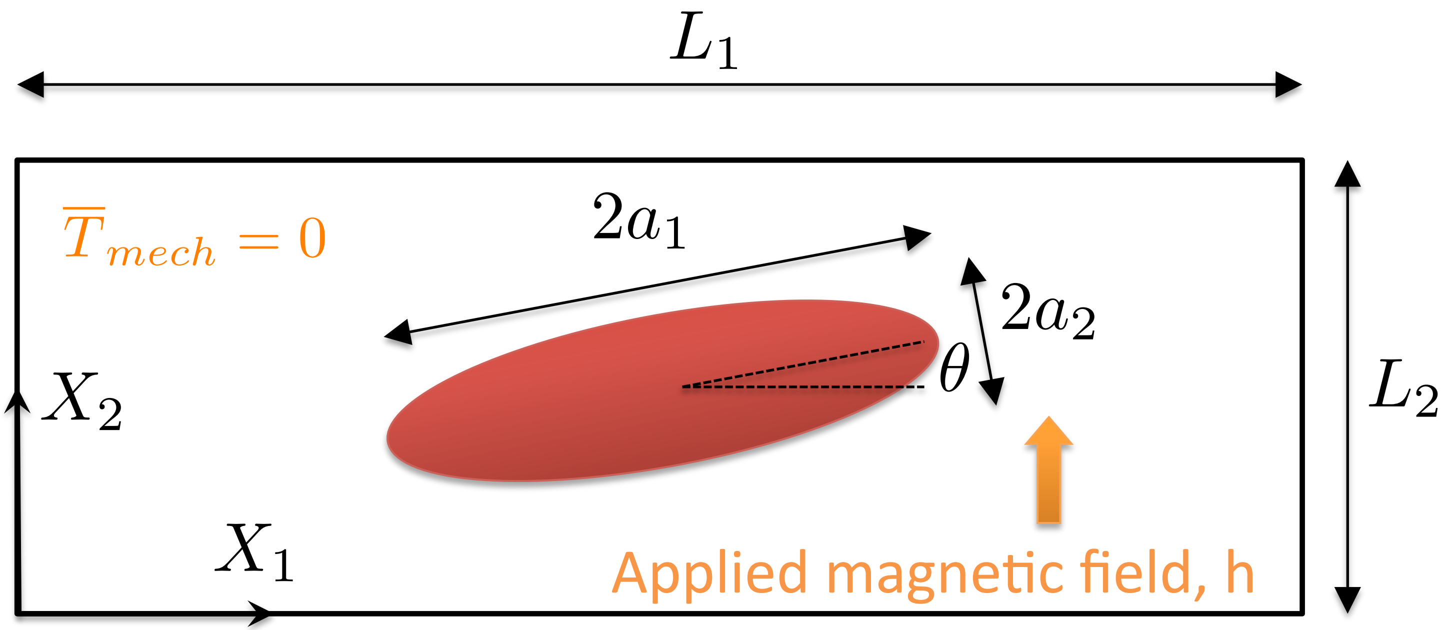
$$\boxed{\overline{\mathbf{T}}_{mech} = \overline{\mathbf{T}}_{tot} - \overline{\mathbf{T}}_{magn} = \overline{\mathbf{T}}_0}$$

- Magnetic potential field

$$\alpha \equiv \text{periodic}$$

(which implies that α should be equal at opposite sides of the unit-cell)

Examples: Shape & Distribution effects

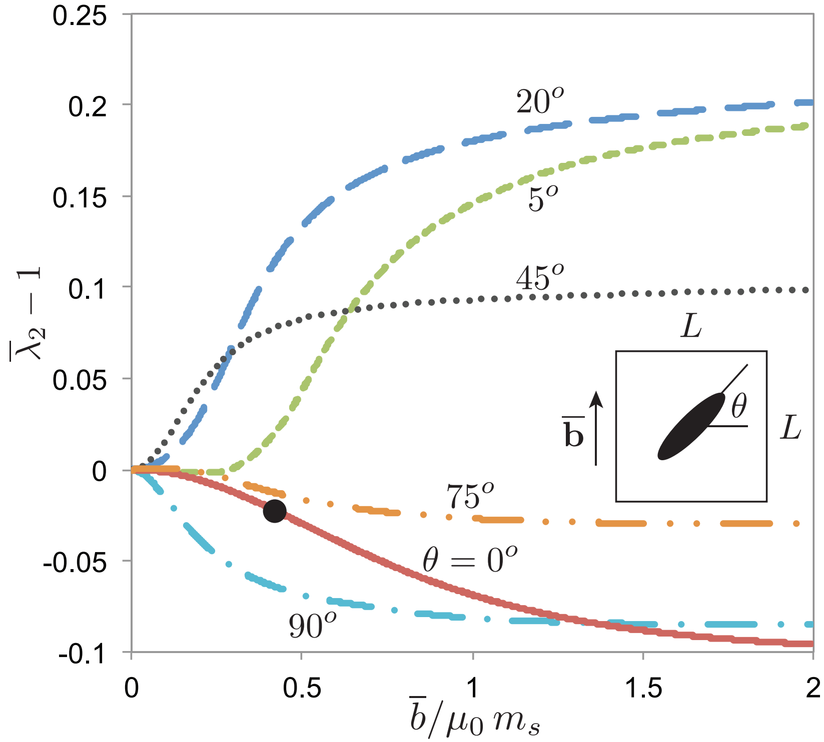


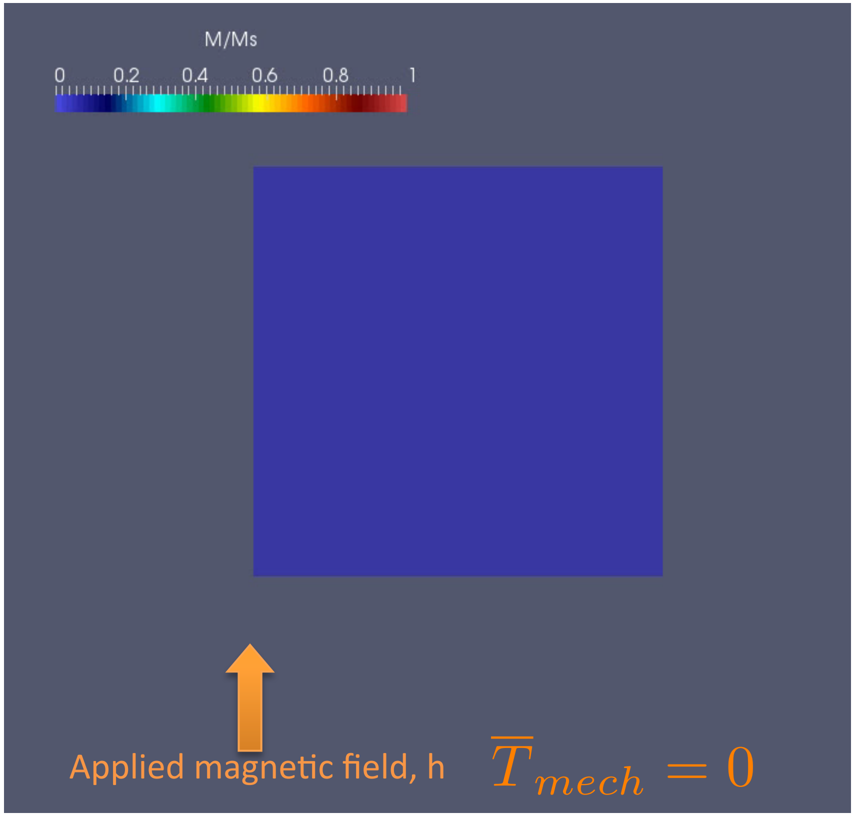
- Particle shape aspect ratio : $w = a_2/a_1$
- Particle orientation : θ
- Unit-cell aspect ratio : $w_d = L_2/L_1$

c=25%, w=0.25, wd=1

$$\overline{F}_{12} = \overline{F}_{21} = 0$$
$$\overline{S}_{11} = \overline{S}_{22} = 0$$

$$\theta = 5^o$$





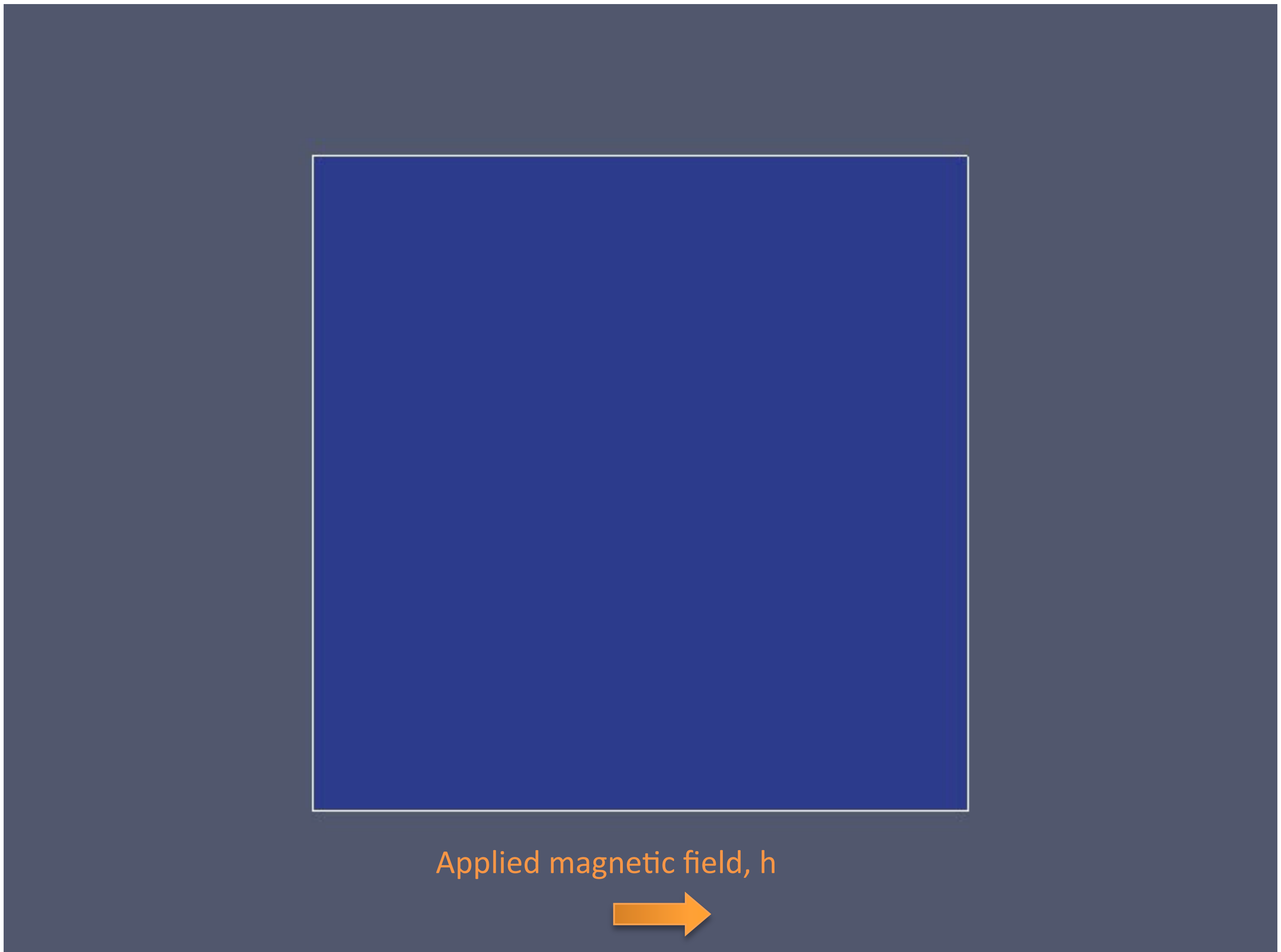
- ⌘ θ=0 leads to no rotation but instability occurs!
- ⌘ A misorientation of the ellipse leads to change in the sign of the macroscopic magnetostriction!

Hierarchical Marginally stable MREs

Danas & Triantafyllidis, JMPS, 2014

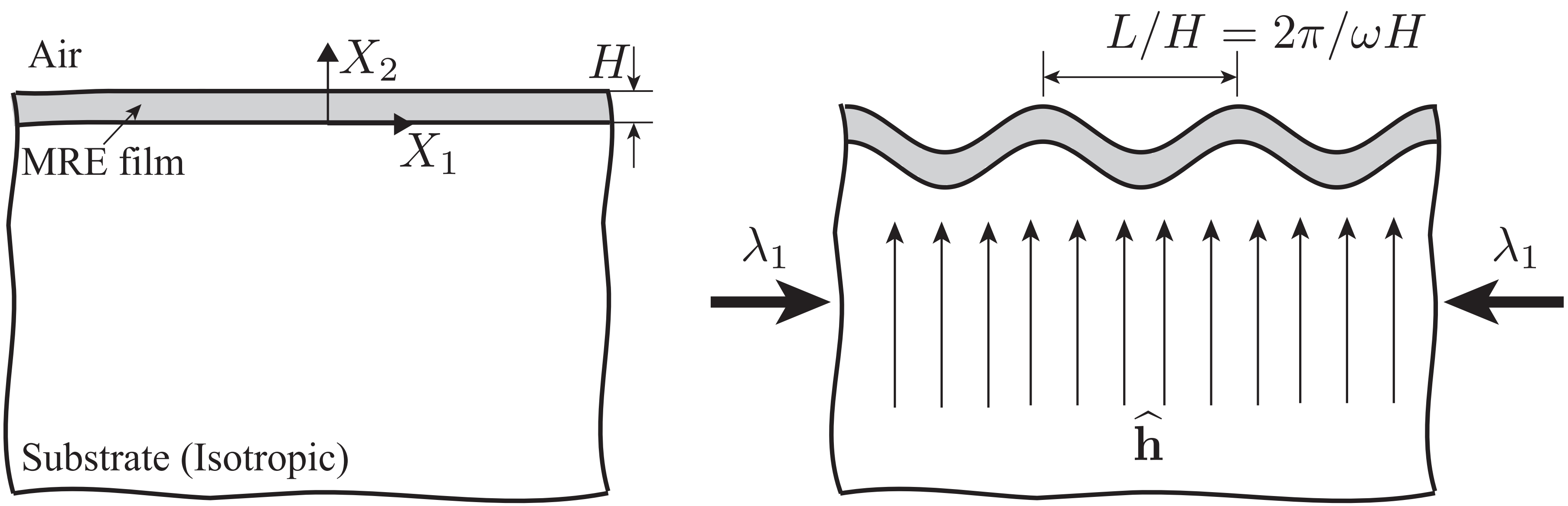
Unstable behavior of MREs

Structural Calculation (no periodicity)



Critically stable macro- & micro-structures

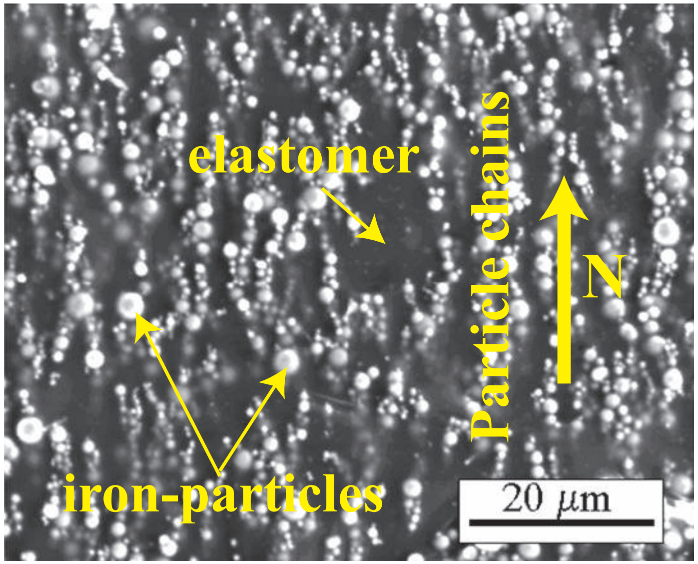
GOAL : buckling at low magnetic fields & for a large range of layer-substrate stiffness ratios



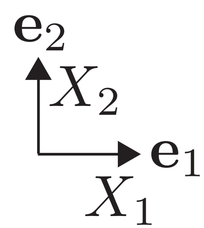
- MRE layer / substrate block
- Magnetic field along X_2
- Axial Compression along X_1
- Infinite layer and substrate in X_1
- Infinite substrate in X_2

Micrographs & Material selection

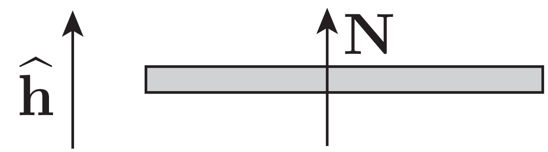
MRE film Microstructure



Ginder et al. 1999, Danas et al., JMPS, 2012



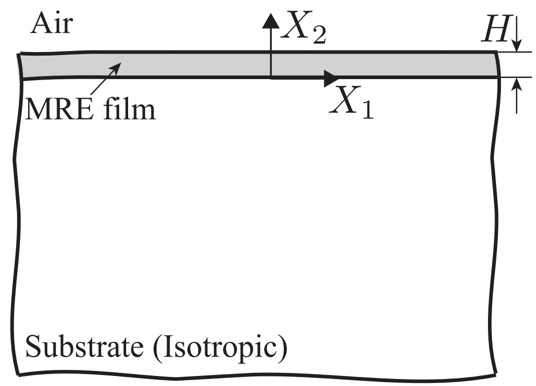
Parallel configuration $(\widehat{\mathbf{h}} \parallel \mathbf{N})$



Perpendicular configuration $(\widehat{\mathbf{h}} \perp \mathbf{N})$

- MRE comprises 25% iron particles of 0.5-5μm size.
- MRE cured in 0.5T magnetic field.
- Curing process leads to formation of particle chains.

An asymptotic analysis



- Energetics of the MRE film

$$\psi_l(\mathbf{F},\mathbf{M}) = G_l\left[\frac{H^2}{12(1-\nu_l)}(w_{,11})^2 + \frac{C_7}{2M_s^2}(\lambda_2 M)^2\right]$$

$$w(X_1) \equiv u_2(X_1, \frac{H}{2}),\; M(X_1) \equiv M_2(X_1, \frac{H}{2})$$

- Energetics of the polymer substrate

$$\psi_s = \frac{G_s\gamma}{2(1-\nu_s)}w^2$$

- Using small strain kinematics :

$$\lambda_2 = 1+\epsilon_{22},\; J = 1+\epsilon_{11}+\epsilon_{22};\quad \epsilon_{22} = -\frac{\nu_l}{1-\nu_l}\epsilon_{11},\; \epsilon_{11} = \frac{1}{2}(w_{,1})^2$$

An asymptotic analysis (Cont.)

- The potential energy of the film/substrate system becomes:

$$\mathcal{P} = \int_0^L \left\{ \frac{G_l H^3}{12(1-\nu_l)}(w_{,11})^2 - \frac{\mu_0 H(\widehat{h})^2}{4(1+\beta)^2}\frac{1-2\nu_l+2\beta\nu_l}{1-\nu_l}(w_{,1})^2 + \frac{\omega\, G_s \gamma}{2(1-\nu_s)} w^2 \right\} \mathrm{d}X_1$$

- The second variation of the potential energy then leads to (Biot, 1937):

$$(\mathcal{P}_{,ww}\Delta w)\delta w = \frac{G_l H^3}{6(1-\nu_l)}\Delta w_{,1111} + \frac{\mu_0 H(\widehat{h})^2}{2(1+\beta)^2}\frac{1-2\nu_l+2\beta\nu_l}{1-\nu_l}\Delta w_{,11} + \frac{\omega G_s \gamma}{1-\nu_s}\Delta w = 0$$

- Substitution of a sinusoidal mode on w, i.e., $\Delta w = W\sin(\omega X_1)$

$$\widehat{h}_c \sim (G_s/G_l)^{1/3} \; ; \quad (\omega H)_c \sim (G_s/G_l)^{1/3}$$

Full Solution and Critical condition

- Second variation of the potential energy:

$$(\mathcal{P}_{,\mathbf{gg}}(\mathbf{g}_0(\boldsymbol{\Lambda}_c), \boldsymbol{\Lambda}_c)\Delta\mathbf{g})\delta\mathbf{g} = 0, \qquad \boldsymbol{\Lambda}_c = \{\lambda_1^c, \widehat{h}_c\}$$

- Look for solution of the type :

$$\Delta u_j(X_1, X_2) = \exp\left(i\omega X_1\right)\left[\sum_I \Delta U_j^I \exp\left(\zeta_I \omega X_2\right)\right],$$

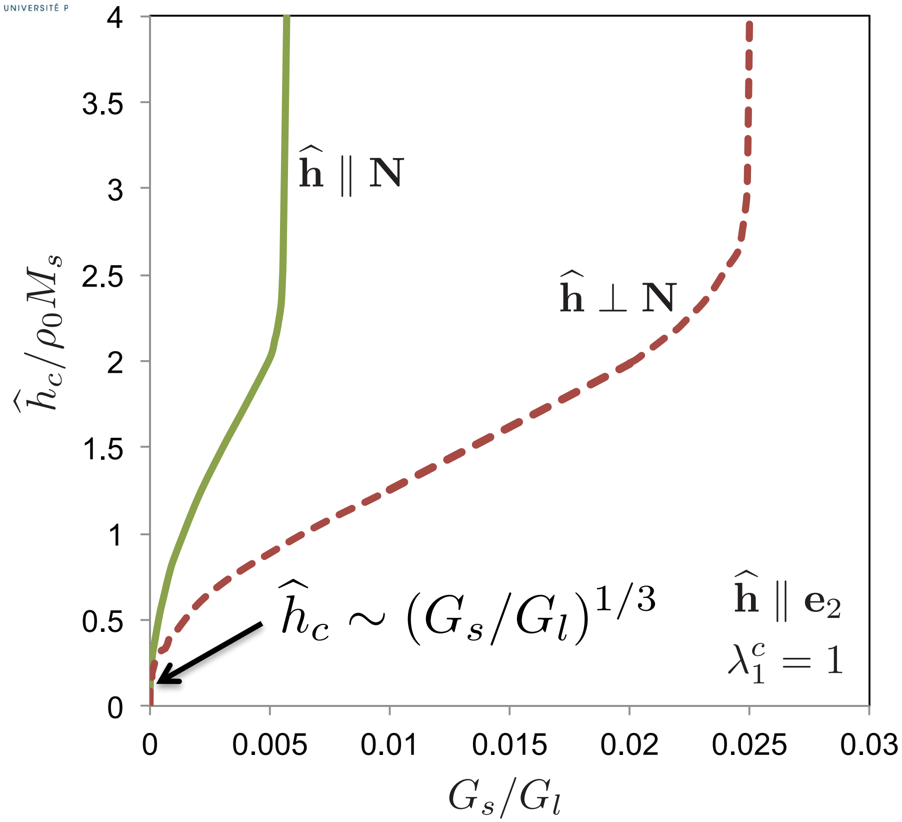
$$\Delta\alpha(X_1, X_2) = \exp\left(i\omega X_1\right)\left[\sum_I \Delta\mathcal{A}^I \exp\left(\zeta_I \omega X_2\right)\right],$$

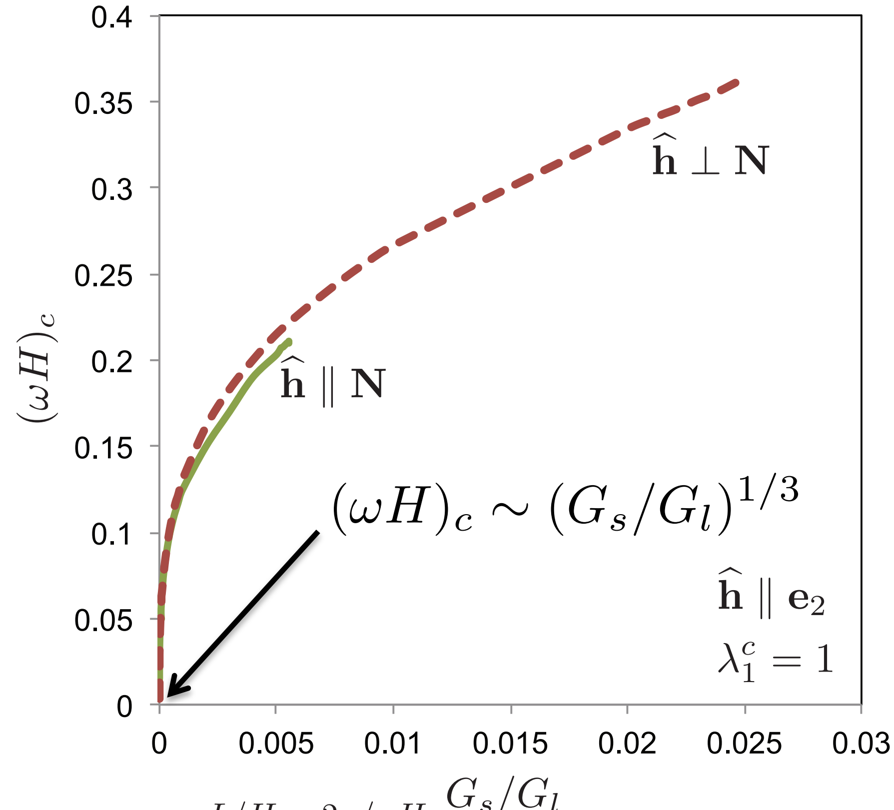
ζ_I : complex roots obtained by the characteristic equation resulting from the Euler-Lagrange equations

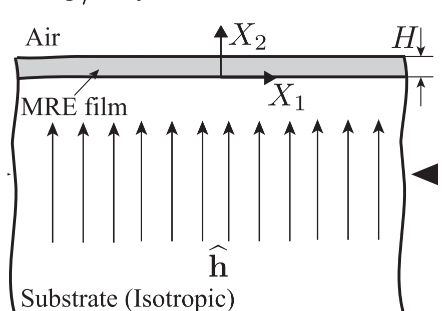
Results

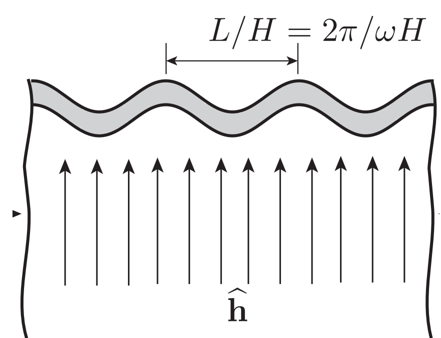
- Purely Magnetic Loading and Zero Stretching
- Purely Mechanical Loading
- Combined Magnetic & Mechanical Loading

Purely Magnetic Loads: Zero stretching



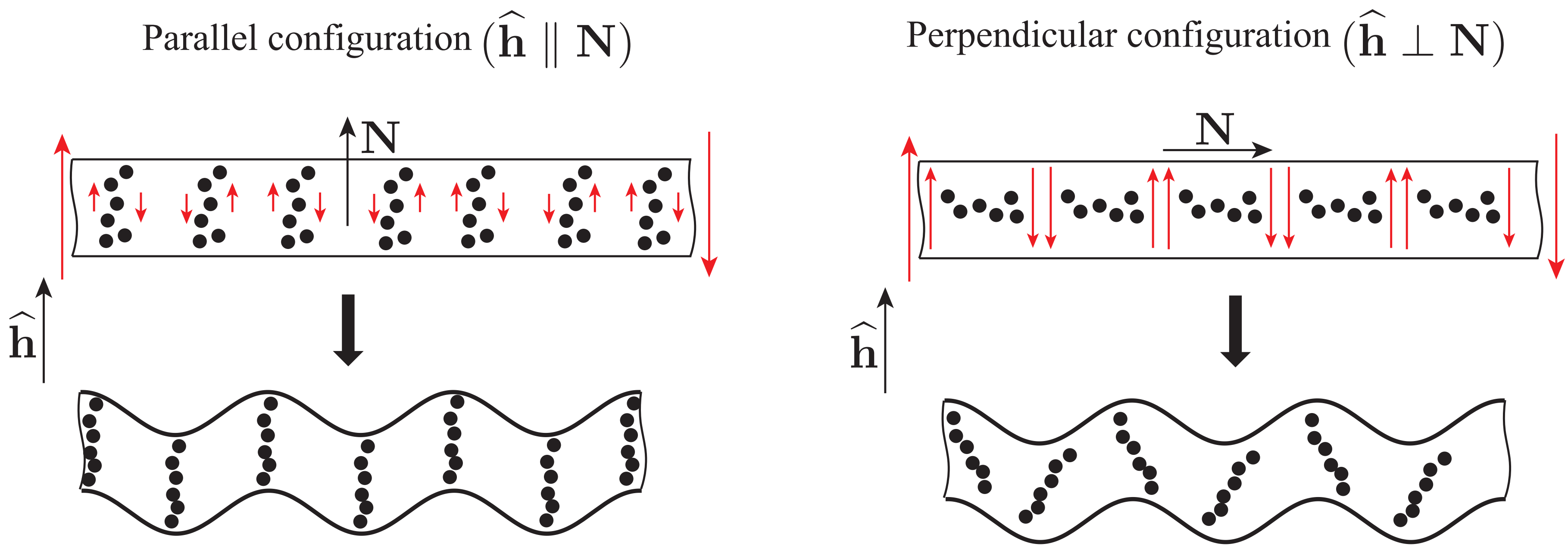






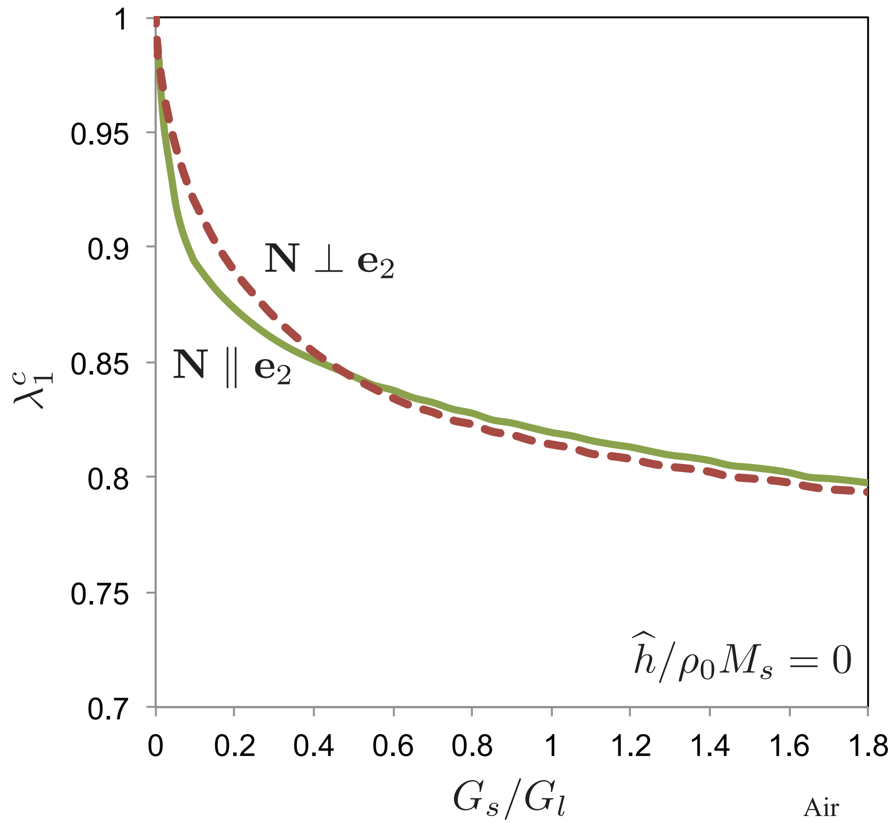
- Shear Modulus of Substrate: G_s
- Shear Modulus of Layer: G_l

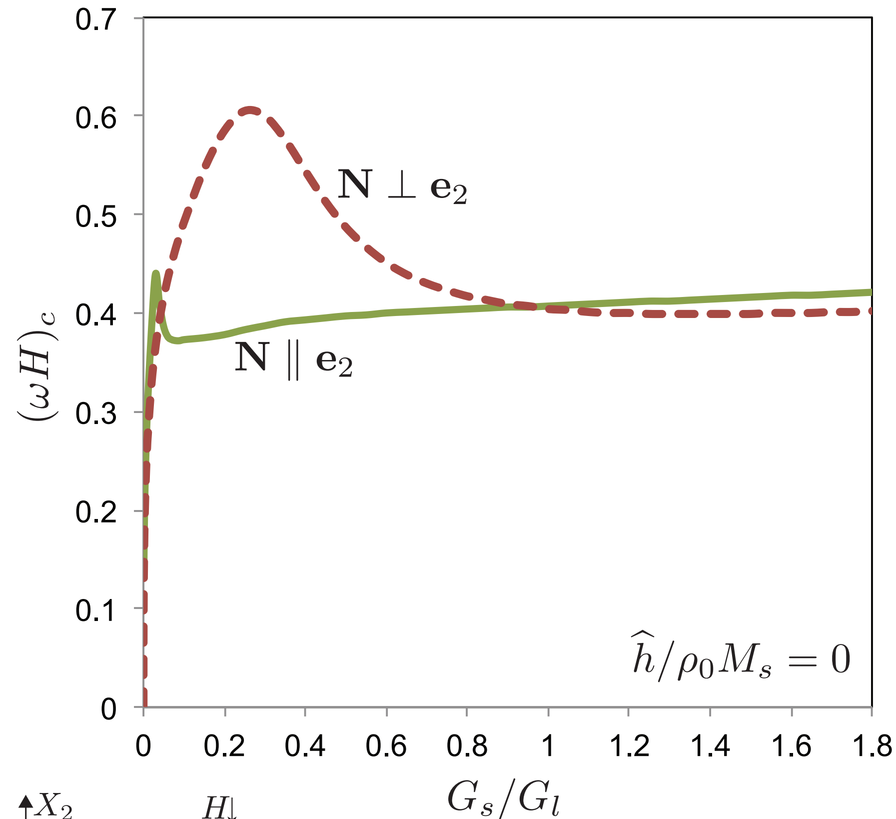
Hierarchical instabilities working together!

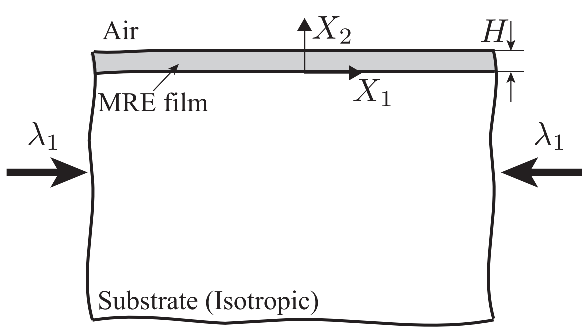


- The instability is a combination of a structural and microstructural instability.
- Hierarchical instabilities explain the difference between Par/Perp
- More studies need to be done at the different scales

Purely Mechanical Loads



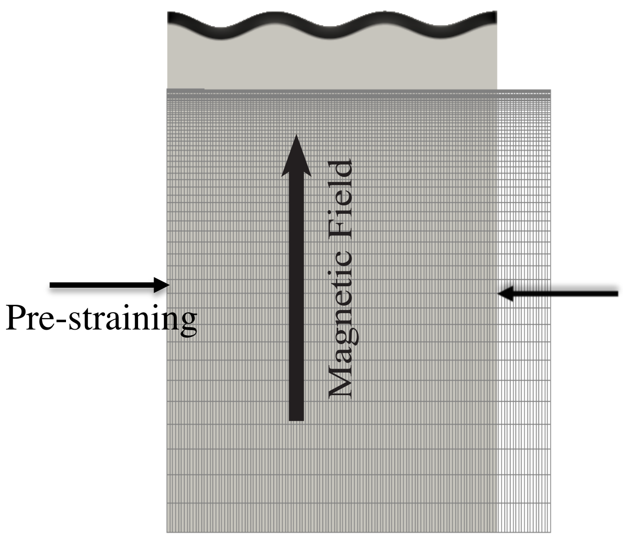


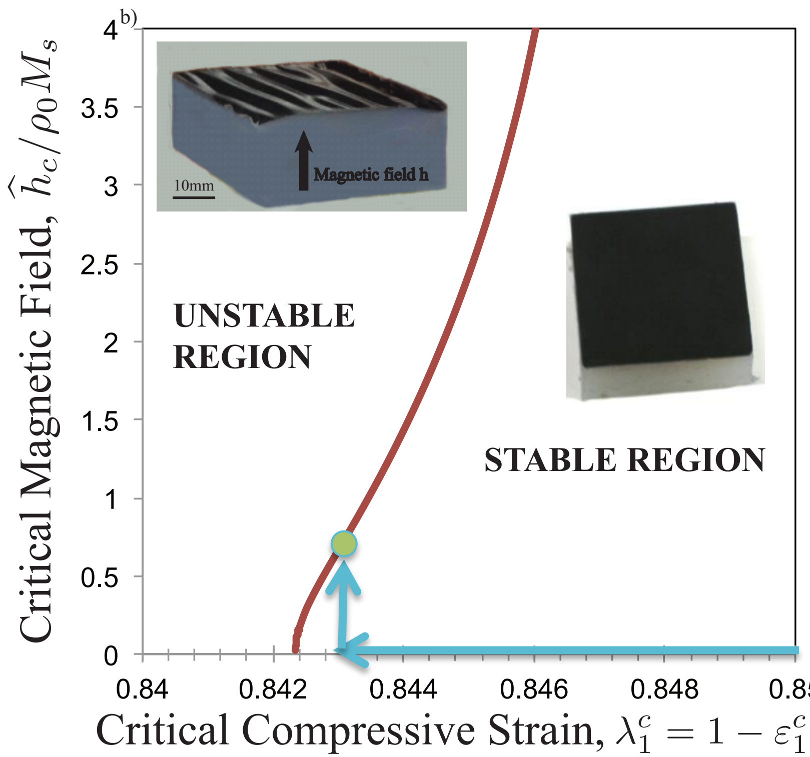


- Shear Modulus of Substrate: G_s
- Shear Modulus of Layer: G_l

Combined Magneto-Mechanical Load : h-λ

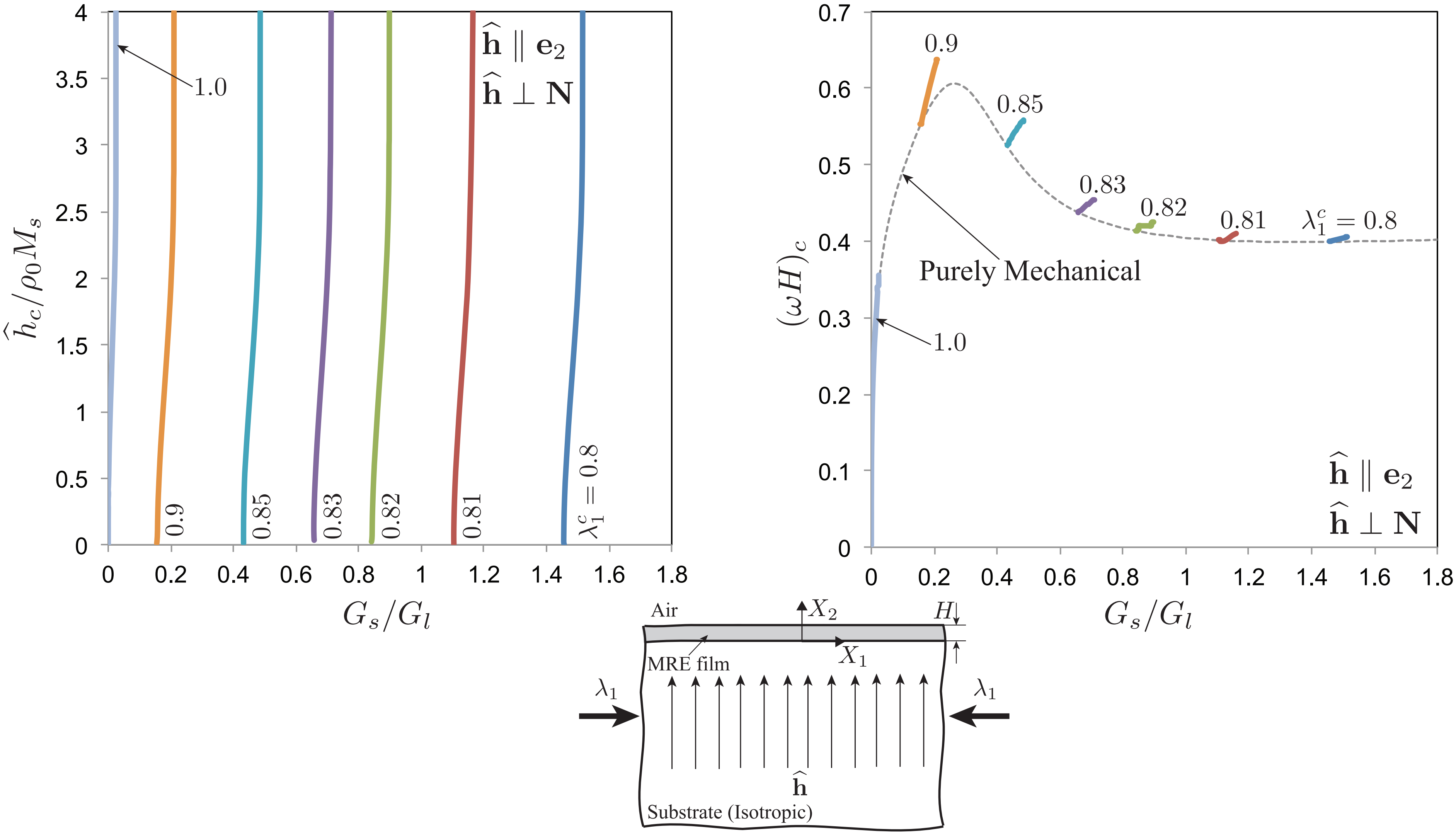
Bring MREs mechanically near a critically stable state → Destabilize with low magnetic field (~0.1Tesla)





Danas & Triantafyllidis, JMPS, 2014

Combined Magneto-Mechanical Load : h-G



- Shear Modulus of Substrate: G_s
- Shear Modulus of Layer: G_l

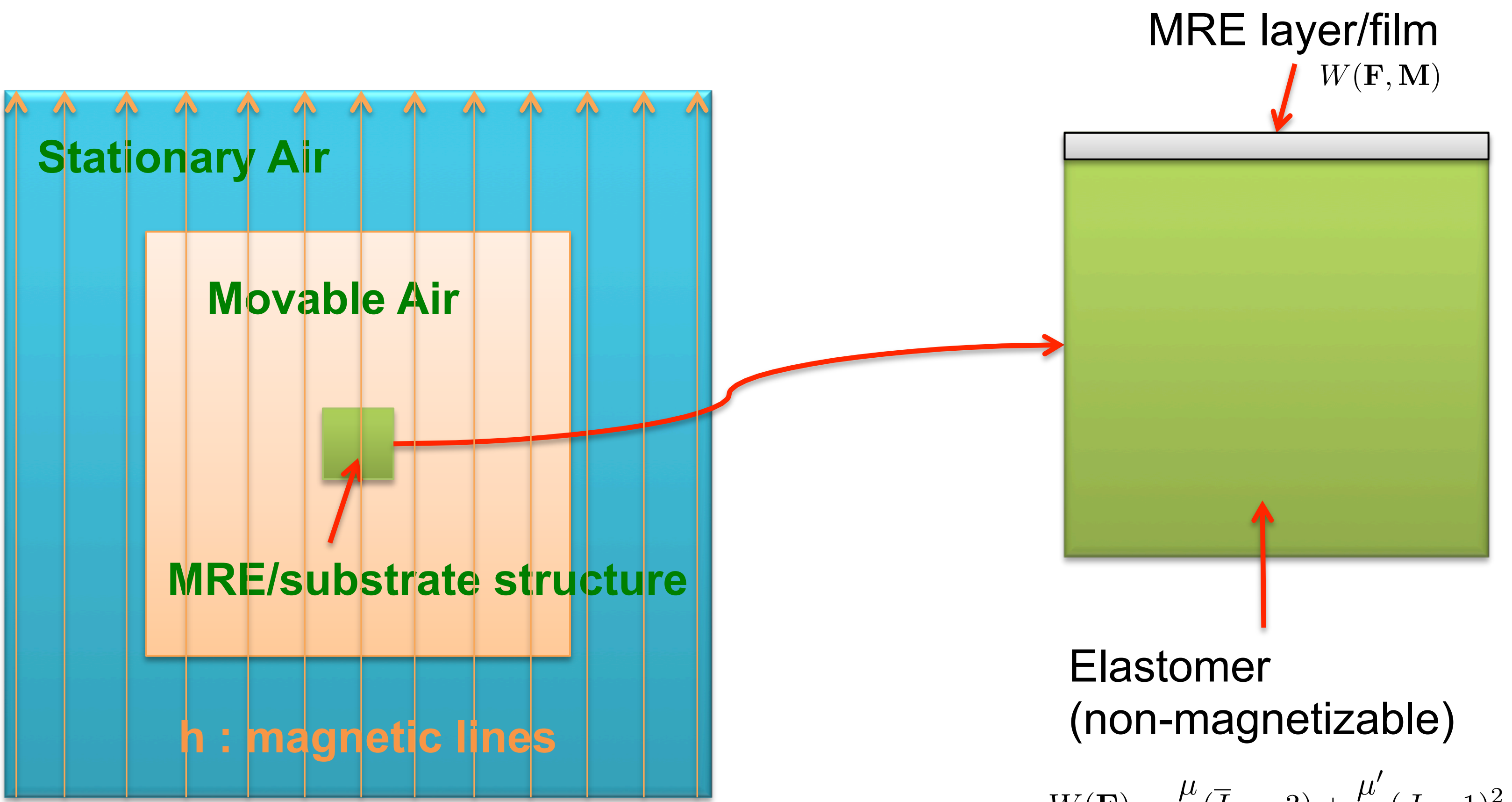
Post-bifurcation with Finite Element

Work in progress

Boundary Value Problems including air

Entire 2D Geometry including both air, MRE layer and substrate

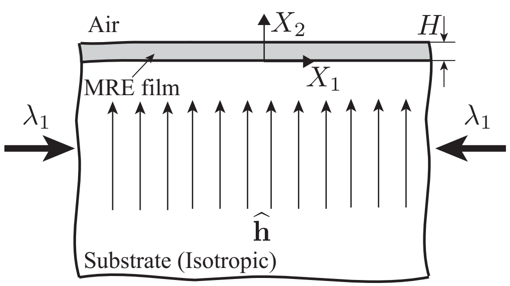
MRE thin layer/film on a passive polymer

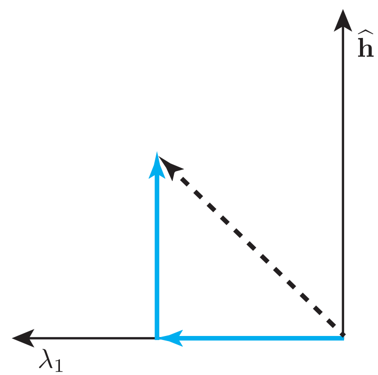


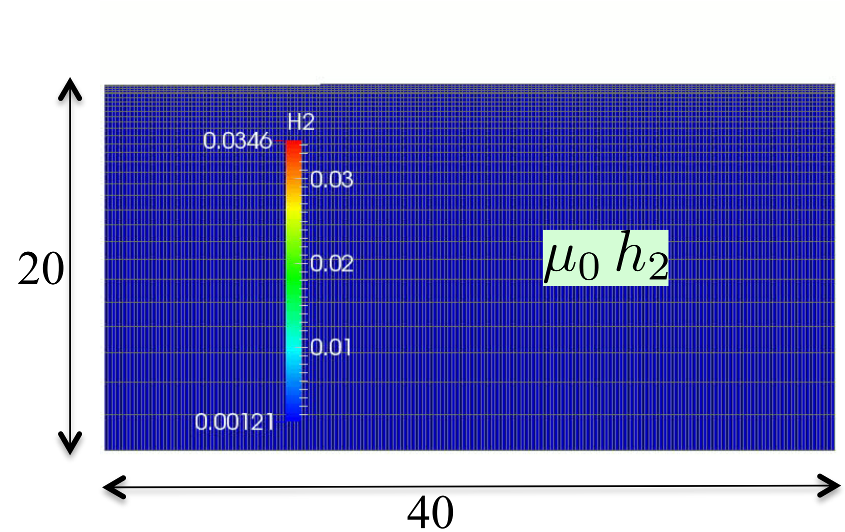
$$W(\mathbf{F}) = \frac{\mu}{2}(\bar{I}_1 - 3) + \frac{\mu'}{2}(J-1)^2$$

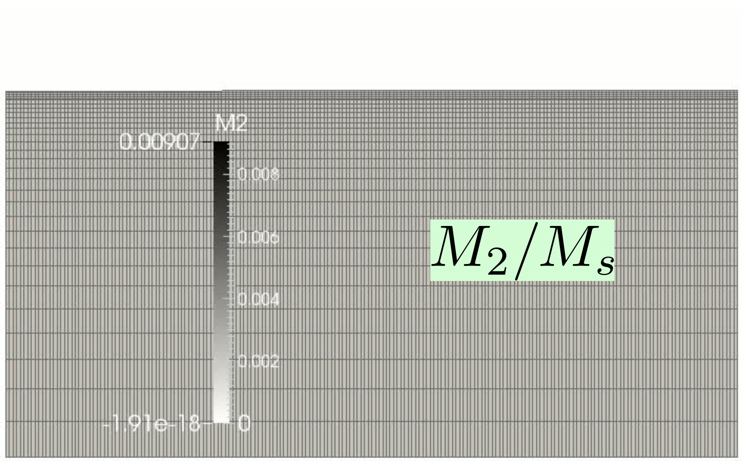
MRE layer / substrate simulation

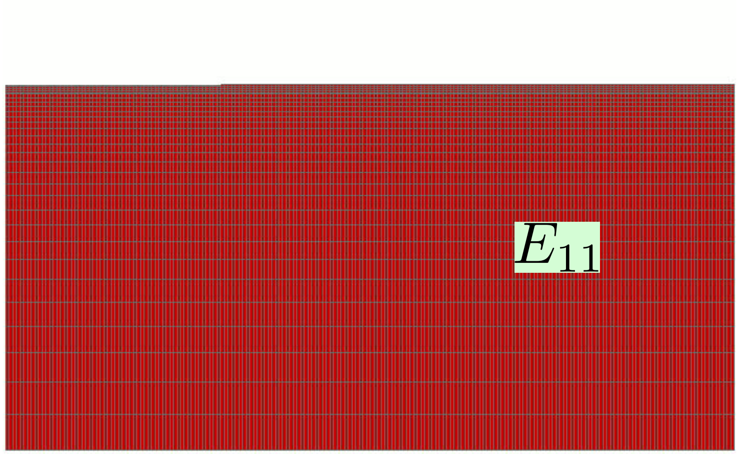
$$G_s/G_l = 0.1$$



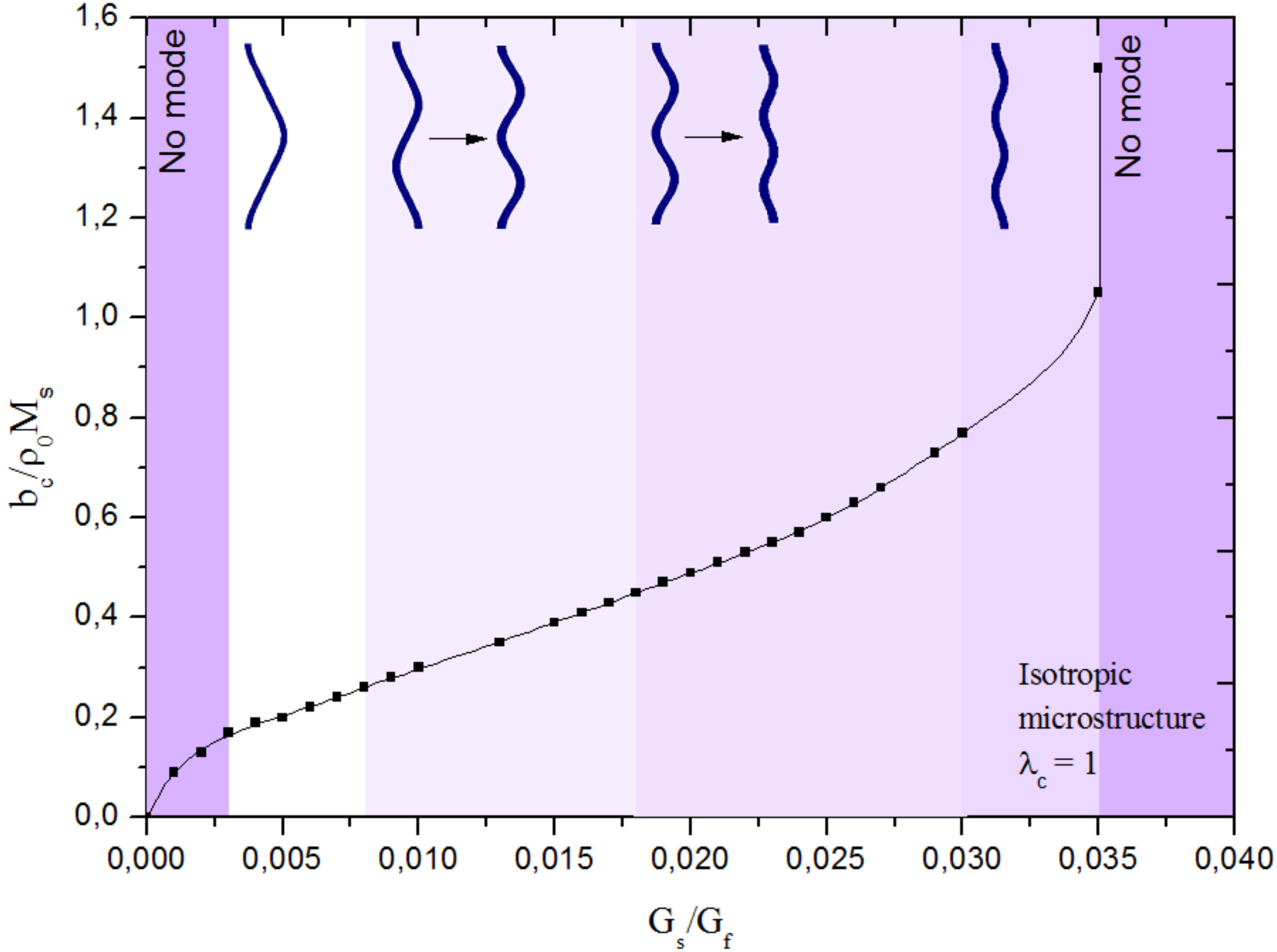




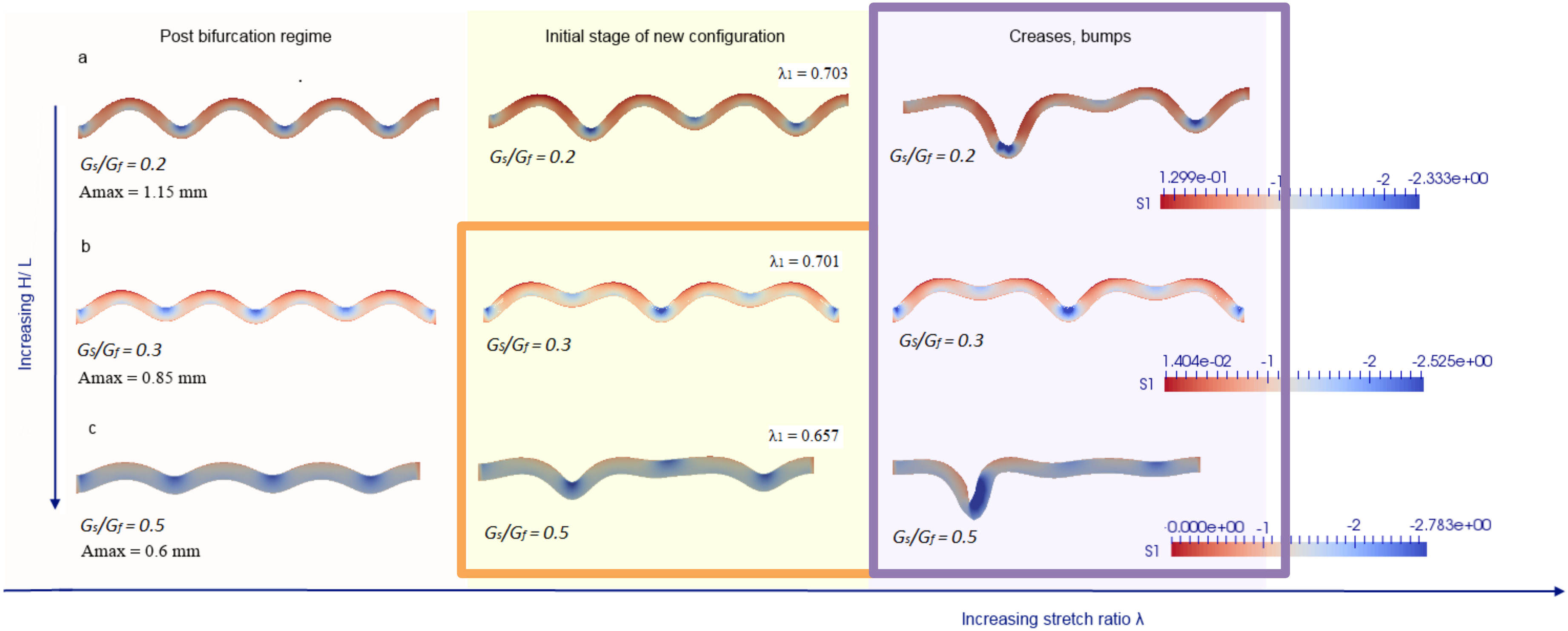




No prestretch : critical field and modes



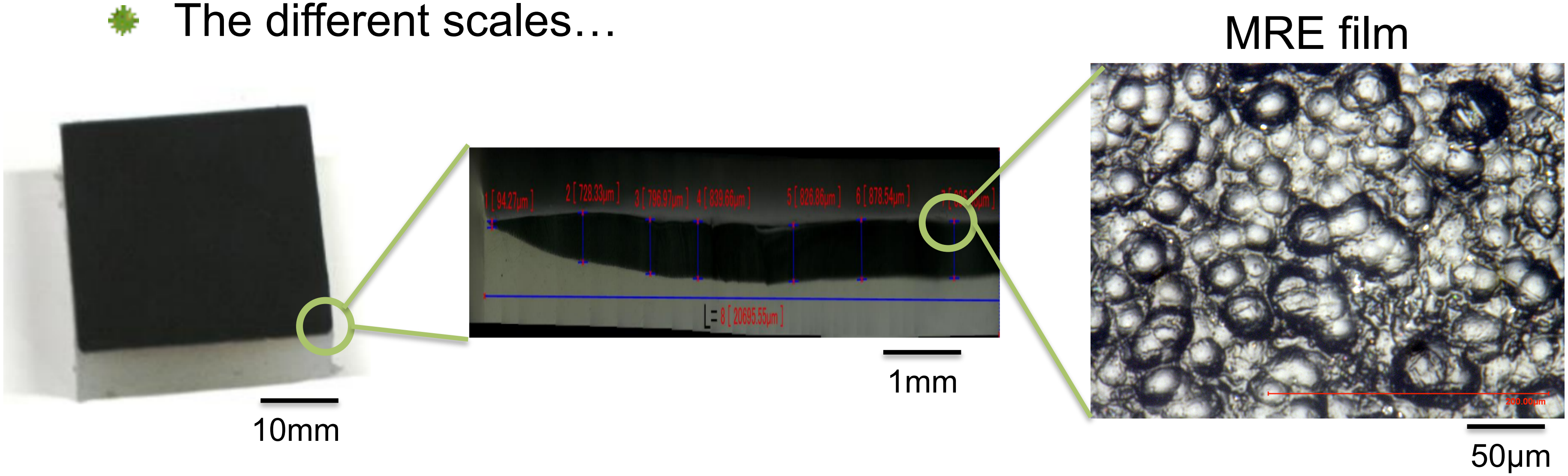
Prestretch, critical field and modes



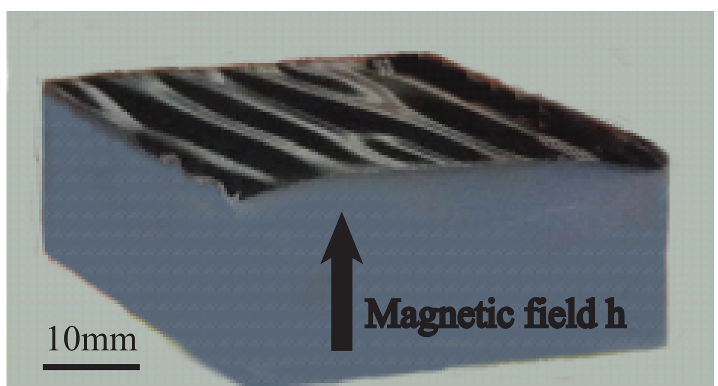
- ⌘ With increasing substrate stiffness & stretch and thickness we form creases…
- ⌘ The interplay of mechanical and magnetic fields is very rich…

Fabrication and Experiments underway…

- The different scales…



- A first experiment…



THANK YOU FOR YOUR ATTENTION

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  ERC-Starting-Grant: MAGNETO

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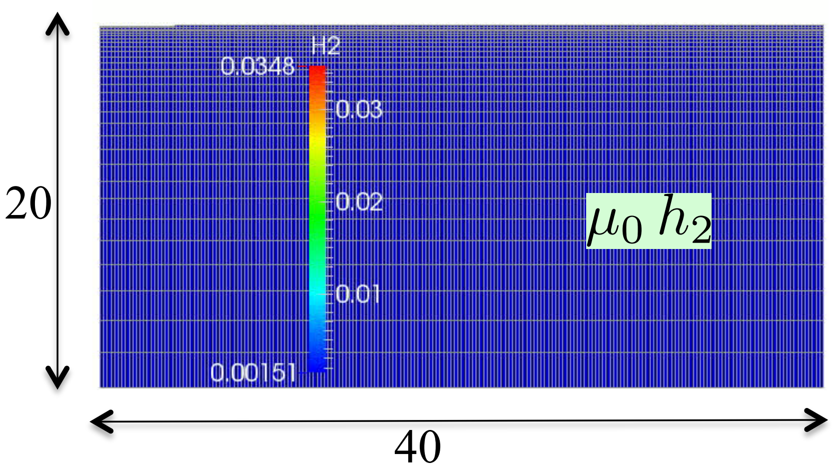
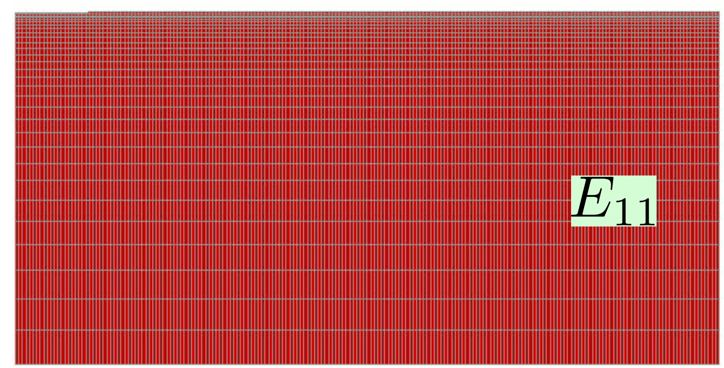
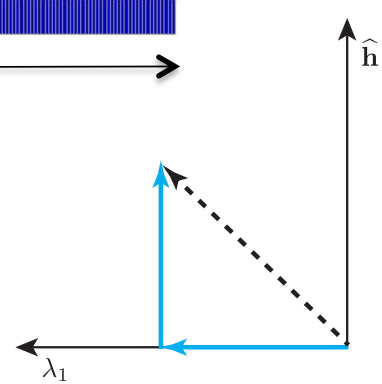
- Ecole Polytechnique
  - http://www.polytechnique.edu
- Laboratory of Solid Mechanics
  - http://www.lms.polytechnique.fr
- Kostas Danas webpage
  - http://hera.polytechnique.fr/users/kdanas

Post-bifurcation response : Ridge or Crease

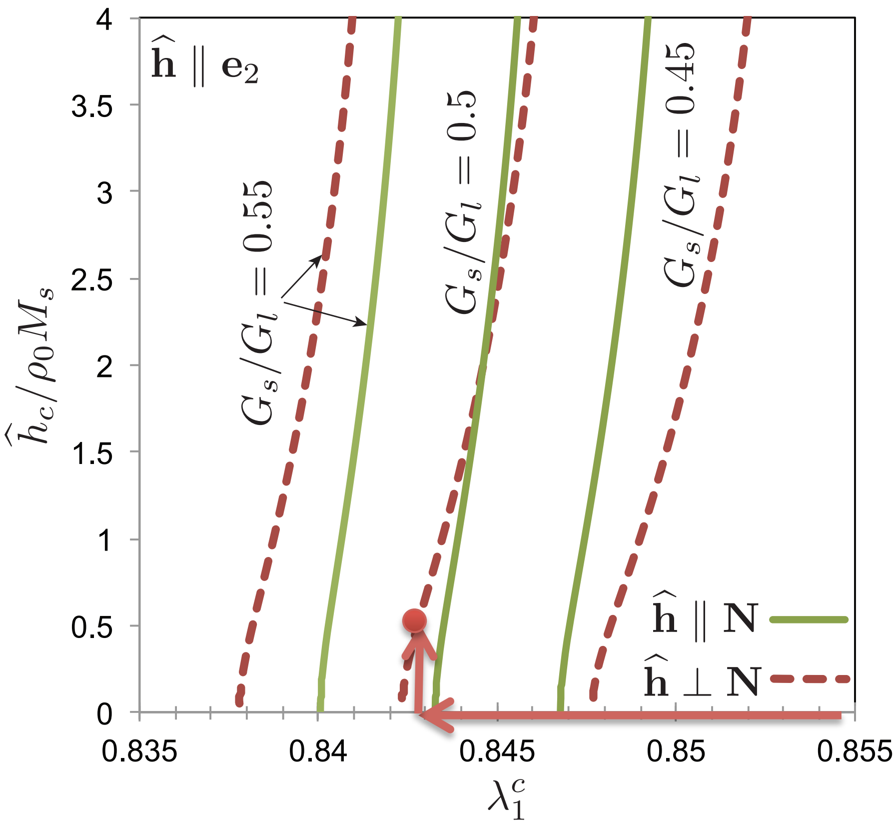
$$G_s/G_l = 0.5$$



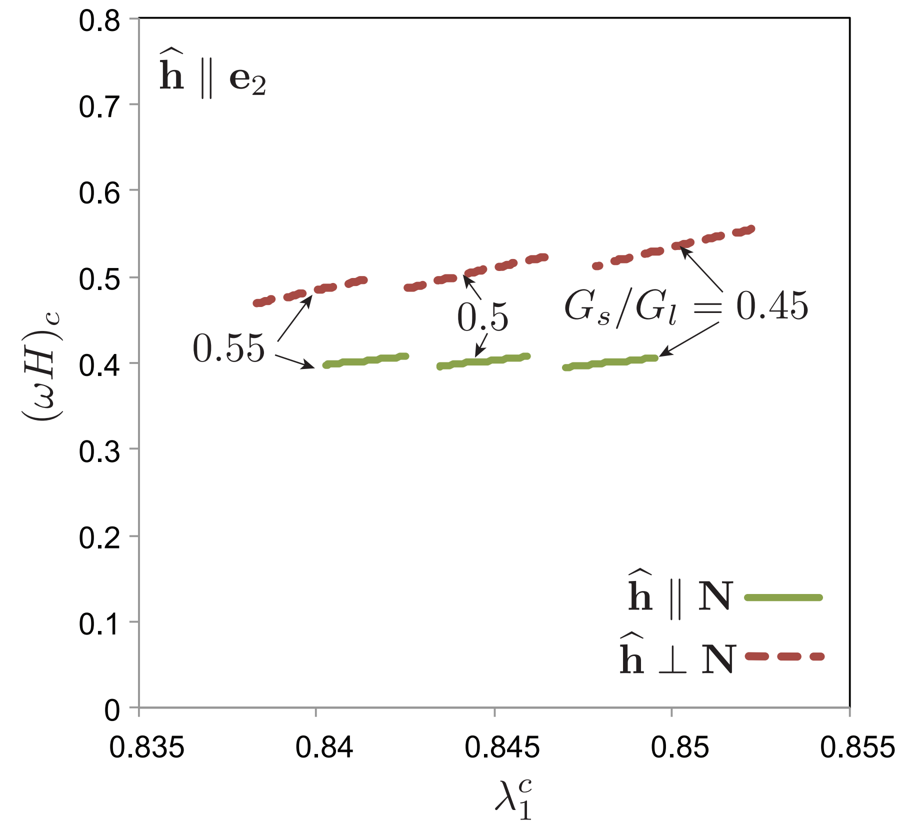




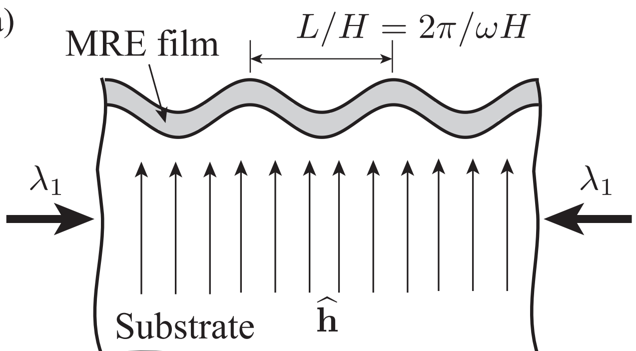
Combined Magneto-Mechanical Load : h-λ

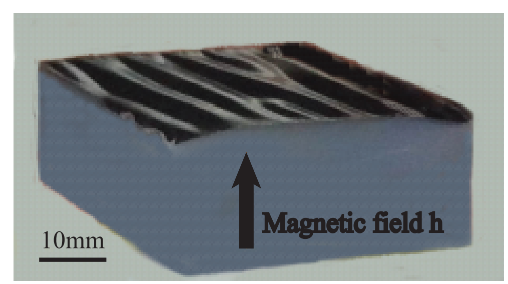


a)

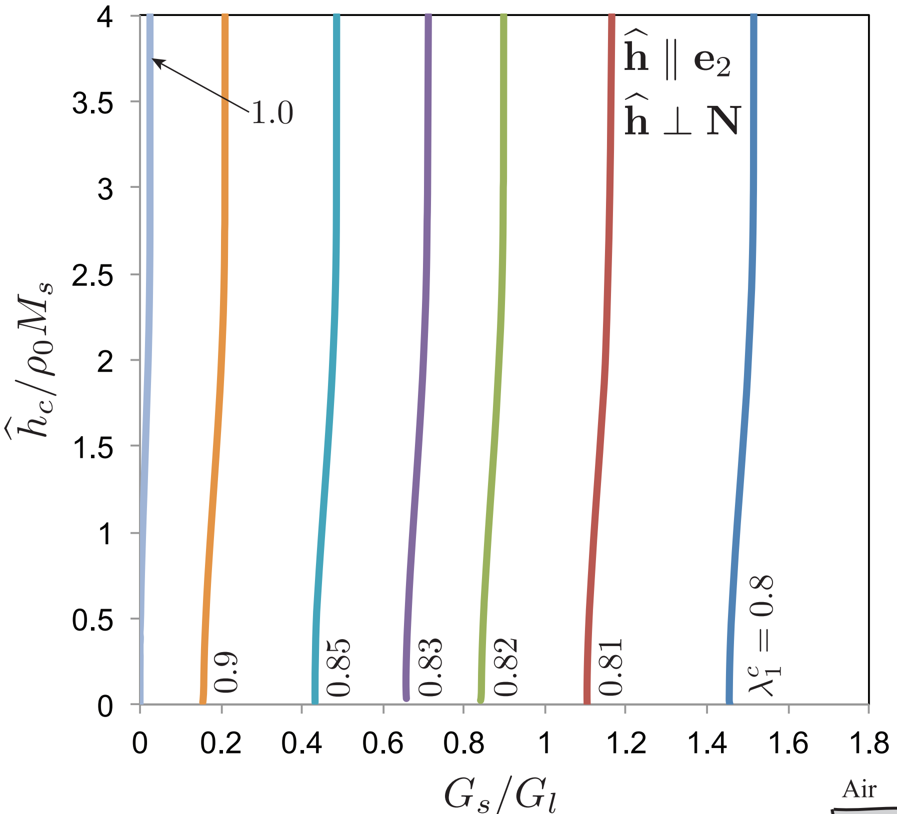


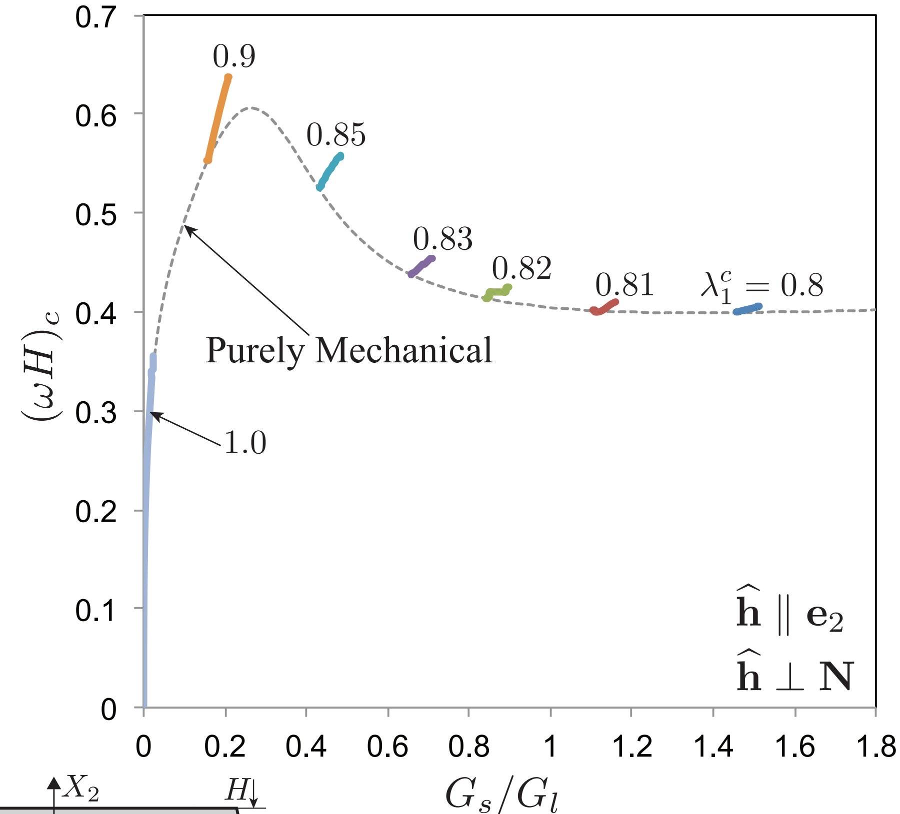
b)

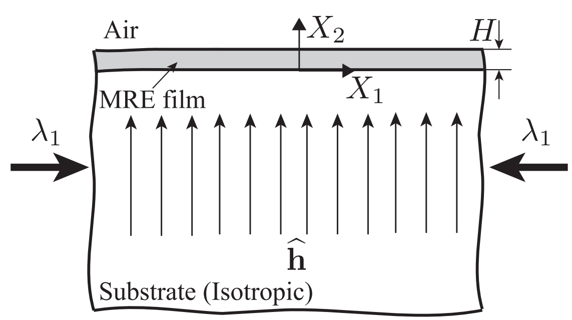




Combined Magneto-Mechanical Load : h-G

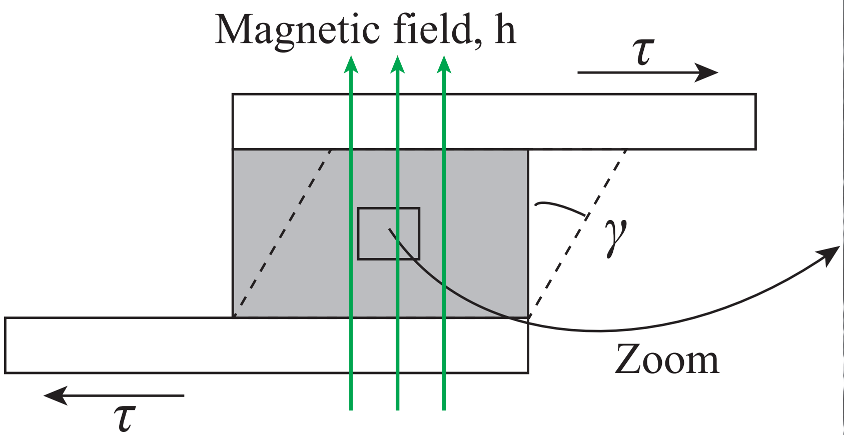




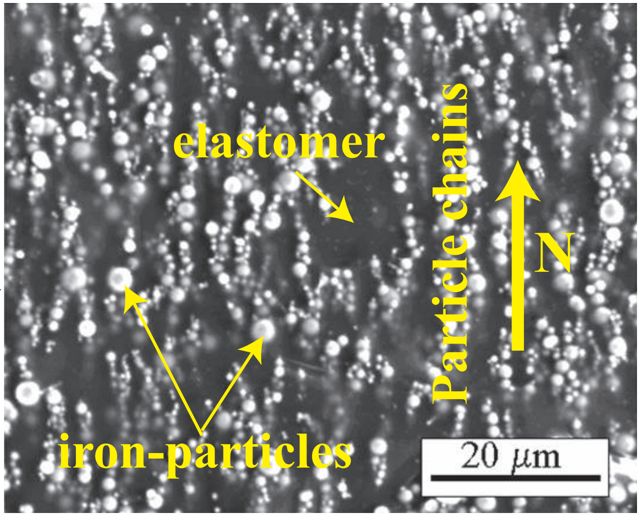


- Shear Modulus of Substrate: G_s
- Shear Modulus of Layer: G_l

MRE Micrographs & Material Fabrication



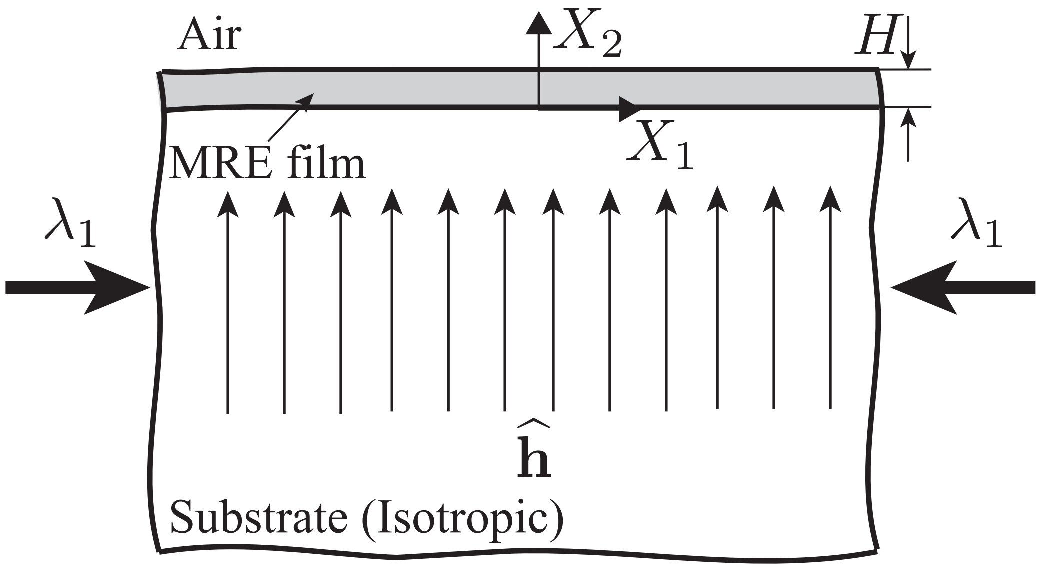
Microstructure



[Ginder et al., 1999]

- MRE comprises 25% iron particles of 0.5-5μm size.
- MRE cured in 0.5T magnetic field.
- Curing process leads to formation of particle chains.

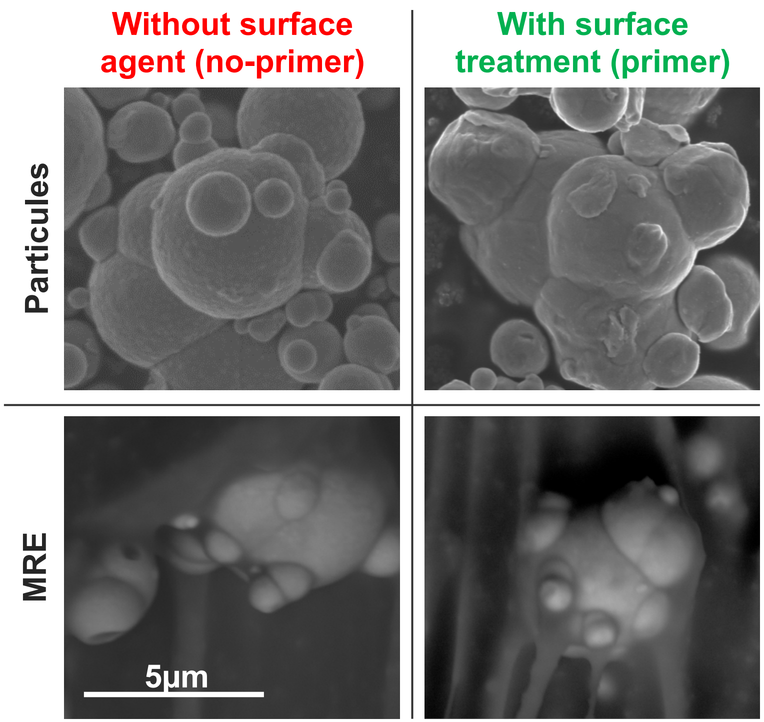
Critically stable MREs



- Combine mechanical and magnetic loads in such a way that:
  - Using first purely mechanical loads bring the system near (but not at) a critically stable regime
  - Trigger the instability with a tiny magnetic field

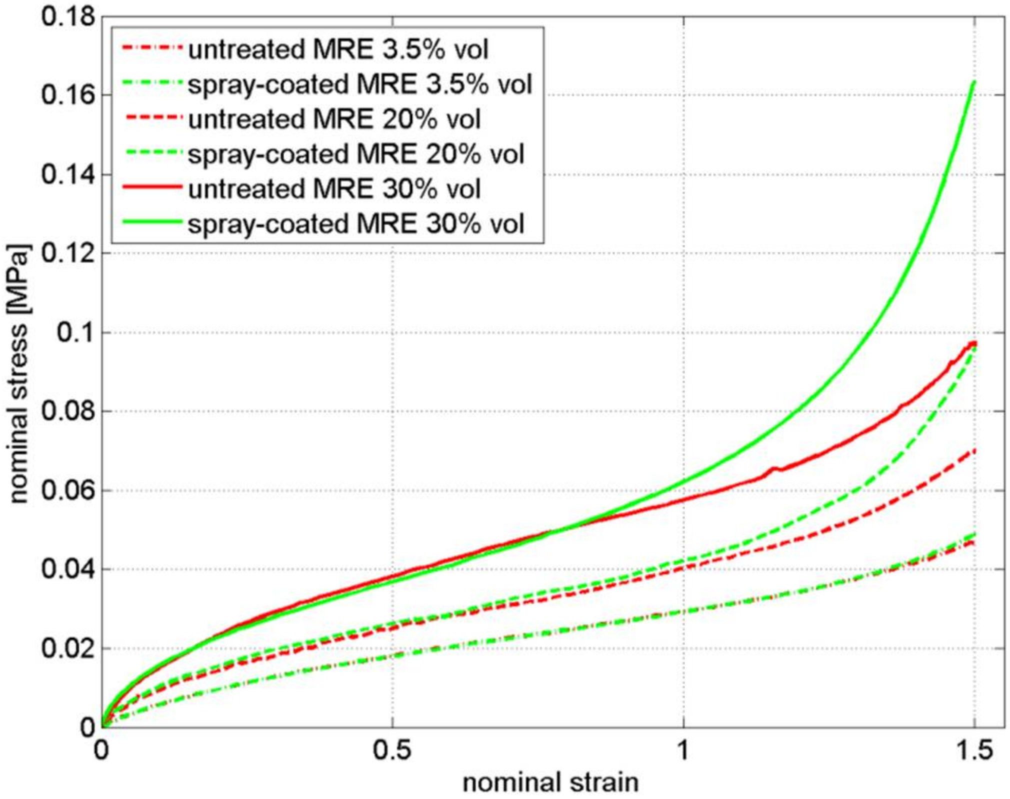
Effect of surface treatment

Isotropic distributions of particles

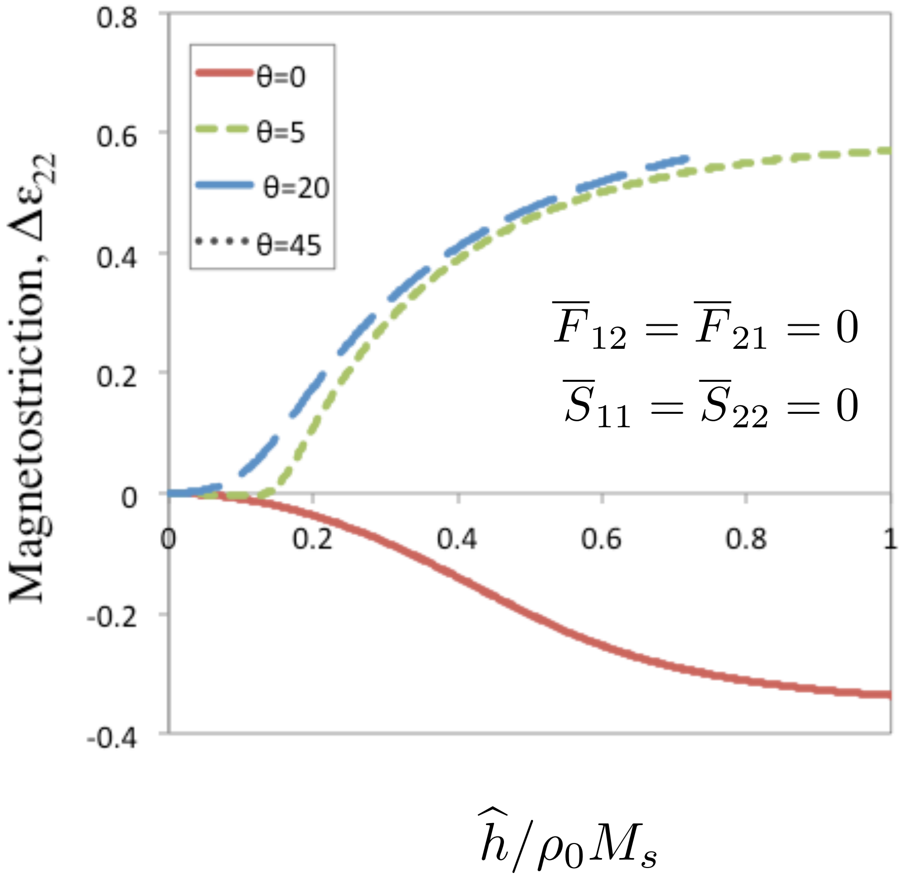


Images taken at the SEM of LMS

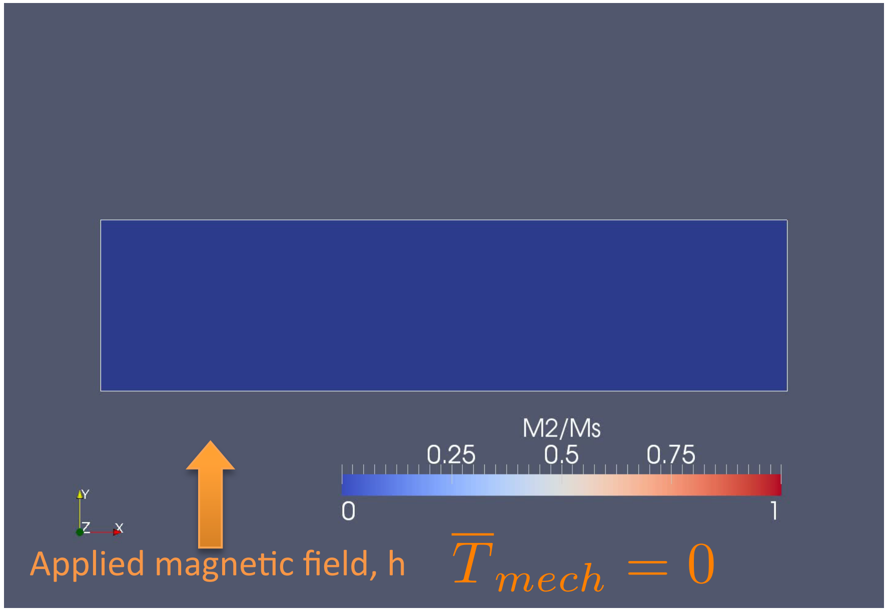
Uniaxial tension at different vol% along the particle chains



c=25%, w=wd=0.25



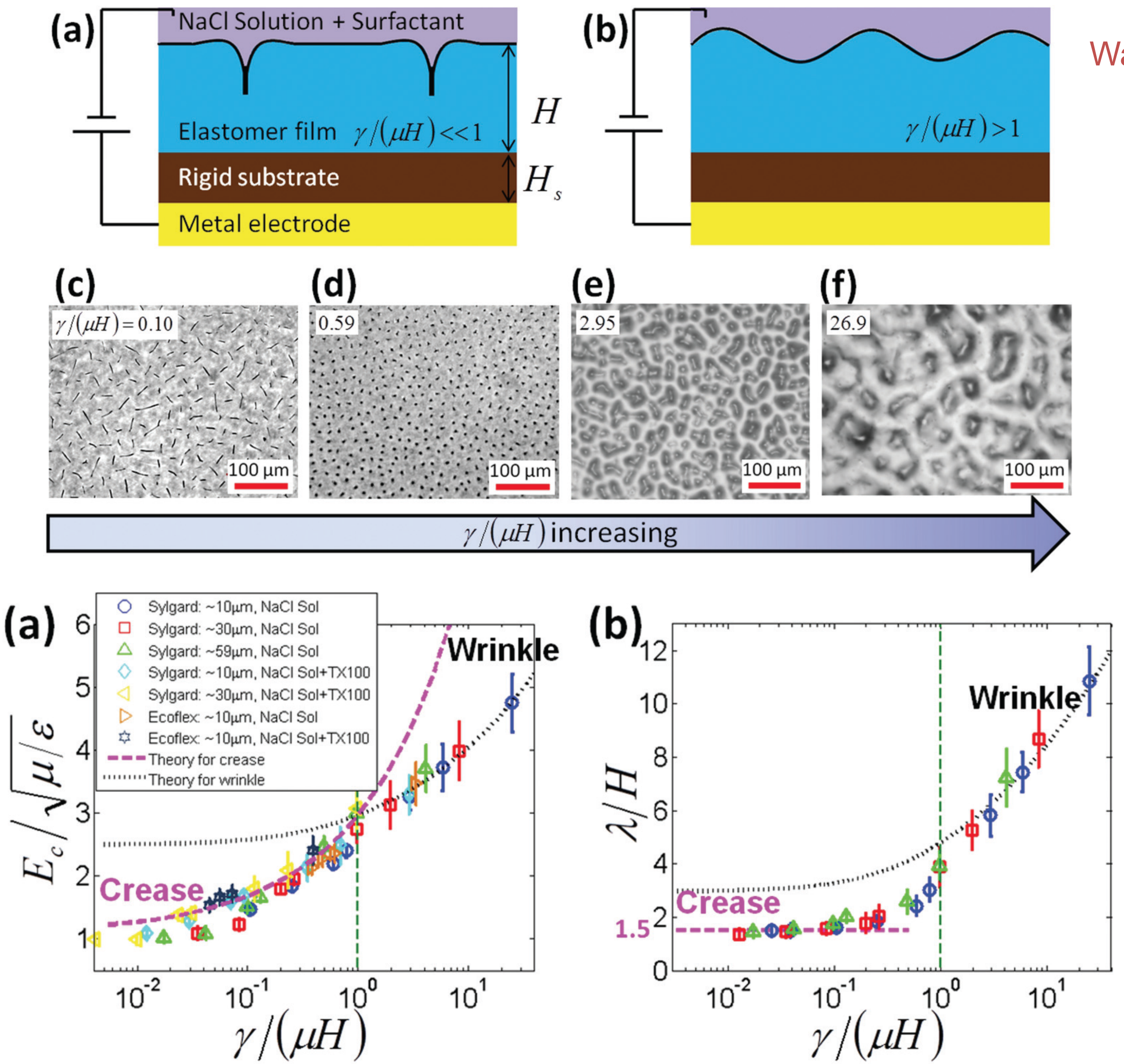
$\theta = 5^o$



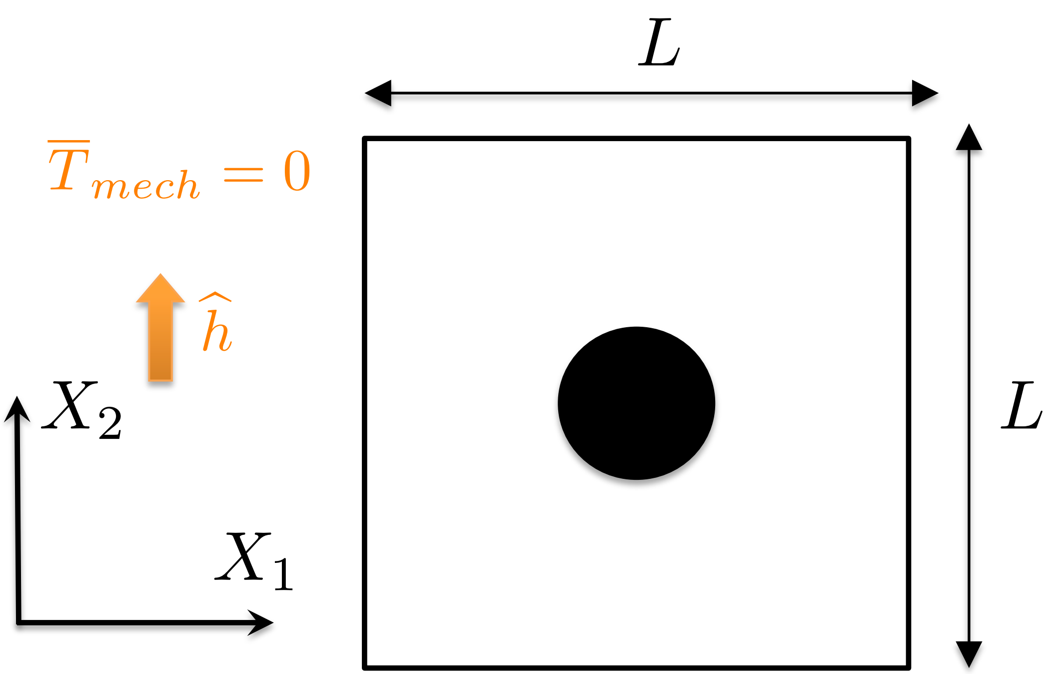
- ⌘ θ=0 leads to no rotation but instability occurs!
- ⌘ A misorientation of the ellipse leads to change in the sign of the macroscopic magnetostriction!

Wrinkling & Creasing with EAPs

Wang & Zhao, PRE, 2013

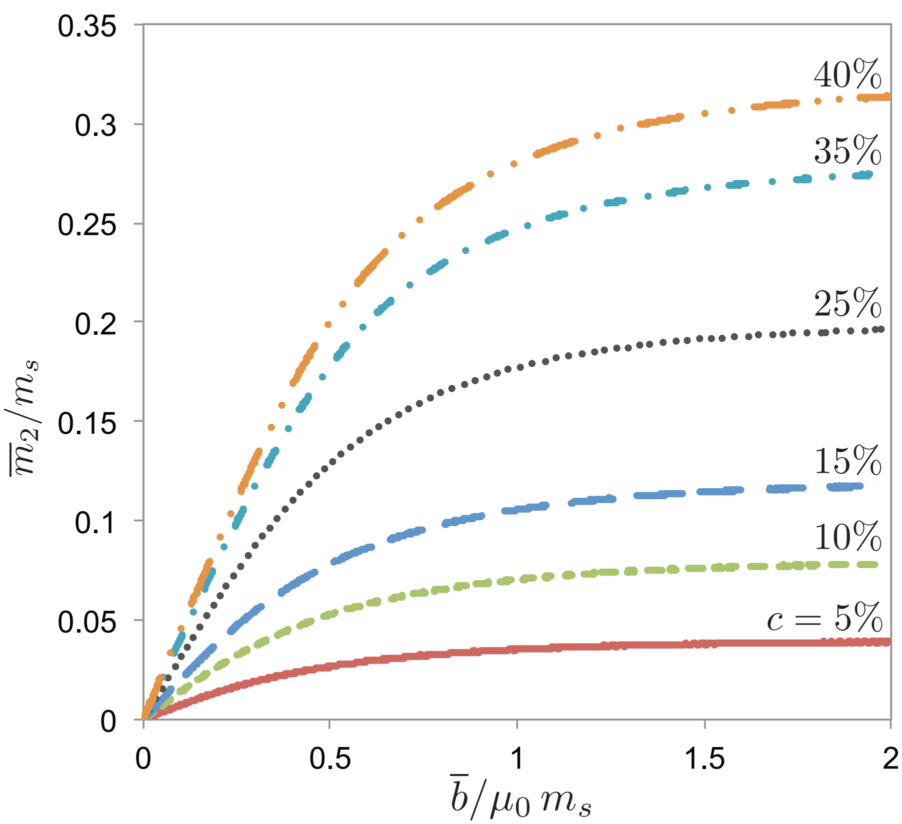


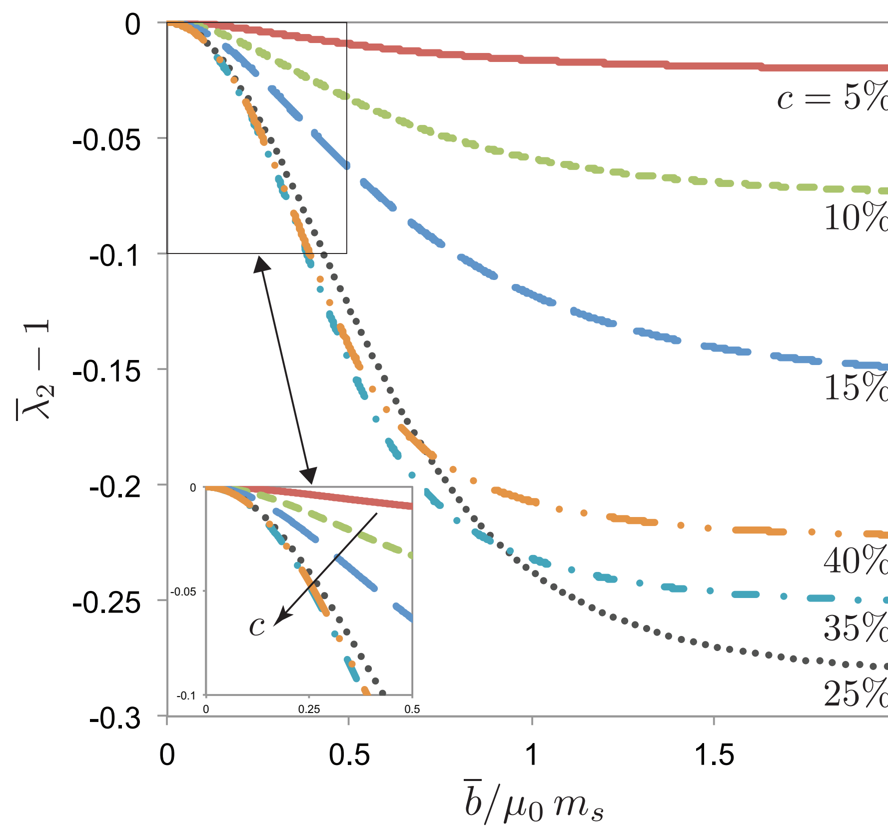
Square distribution



- Particle vol % : 5-35%
- Number of particles : 1

c=25%, w=wd=1

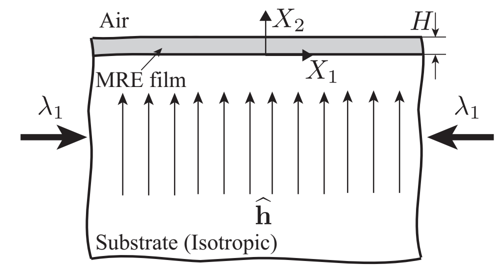




- ⌘ Monotonic dependence on vol% for small fields
- ⌘ Non-monotonic dependence on vol% for large fields due to magneto-mechanical interplay at finite strains

Interface/Boundary Conditions

- Interface/Boundary conditions at :



Essential $\quad (X_1, X_2) \in \mathbb{R} \times \{0\} \quad$ Natural

$$\{\Delta u_i\}_l = \{\Delta u_i\}_s$$

$$\{\Delta \alpha\}_l = \{\Delta \alpha\}_s$$

$$\{L^{uu}_{i2kl}\Delta u_{k,l} + L^{u\alpha}_{i2k}\Delta \alpha_{,k}\}_l = \{L^{uu}_{i2kl}\Delta u_{k,l} + L^{u\alpha}_{i2k}\Delta \alpha_{,k}\}_s$$

$$\left\{L^{\alpha u}_{2jk}\Delta u_{j,k} + L^{\alpha\alpha}_{2j}\Delta \alpha_{,j}\right\}_l = \left\{L^{\alpha u}_{2jk}\Delta u_{j,k} + L^{\alpha\alpha}_{2j}\Delta \alpha_{,j}\right\}_s$$

- Interface/Boundary conditions at :

Essential $\quad (X_1, X_2) \in \mathbb{R} \times \{H\} \quad$ Natural

$$\{\Delta \alpha\}_l = \{\Delta \alpha\}_a$$

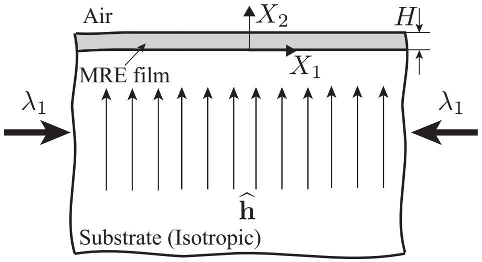
$$\{L^{uu}_{i2kl}\Delta u_{k,l} + L^{u\alpha}_{i2k}\Delta \alpha_{,k}\}_l = 0$$

$$\left\{L^{\alpha u}_{2jk}\Delta u_{j,k} + L^{\alpha\alpha}_{2j}\Delta \alpha_{,j}\right\}_l = 0$$

NOTE: Incremental moduli: $L^{uu}_{ijkl} = \dfrac{\partial^2 \mathcal{P}}{\partial F_{ij}\partial F_{kl}}, L^{u\alpha}_{ijk} = \dfrac{\partial^2 \mathcal{P}}{\partial F_{ij}\partial \widetilde{A}_k}, L^{\alpha\alpha}_{ij} = \dfrac{\partial^2 \mathcal{P}}{\partial \widetilde{A}_i\partial \widetilde{A}_j}$

Critical Condition and critical load

- The use of Euler-Lagrange equations together with the boundary conditions lead to 10x10 system:



$$\sum_{q=1}^{10} \mathcal{D}_{pq}(\mathbf{\Lambda}, \omega H)\, \mathcal{H}_q = 0, \quad 1 \leq p \leq 10$$

where

$$\mathcal{H}_q = \mathcal{H}_q(\Delta U^I, \Delta \mathcal{A}^I)$$

- Critical condition and critical load:

$$\boxed{\det[\boldsymbol{\mathcal{D}}(\mathbf{\Lambda}, \omega H)] = 0, \qquad \mathbf{\Lambda}_c = \{\lambda_1^c, \widehat{h}_c\} = \inf_{\omega H} \mathbf{\Lambda}(\omega H)}$$

⌘ Interesting to see what Catastrophe Theory can tell us?